# Stochastic hyperviscous Navier-Stokes equations

B. Ferrario

Università degli Studi di Pavia- Italy

*BIRS workshop "Regularity and blow-up of Navier-Stokes type PDEs using harmonic and stochastic analysis'*

# Outline

1. stochastic Navier-Stokes equations
2. hyperviscous stochastic Navier-Stokes equations
3. well posedness
4. analysis of the law (representation formula by means of Girsanov transform)
    - working on the velocity equation
    - working on the vorticity equation
5. some Gaussian invariant measures (and open problems)

$$\begin{cases} \partial_t v + [-\nu\Delta v + (v\cdot\nabla)v + \nabla p]\ dt = G\ dw \\ \nabla\cdot v = 0 \end{cases} \tag{1}$$

where $v = \vec{v}(t,\vec{\xi}), p = p(t,\vec{\xi})$ with $t \geq 0, \vec{\xi} \in D \subset \mathbb{R}^d$ for $d = 2, 3$

$+$ initial condition

$+$ boundary condition

Why to add a stochastic term?

numerical and empirical uncertainties $\rightsquigarrow$ Gaussian model

Main issues for a stochastic PDE

- well posedness (for weak or strong solutions)
- analysis of the law (the solution is a stochastic process):
  - invariant measures ($\exists$ and !)
  - asymptotic behaviour (as $t \to \infty$)

⋮

$$\begin{cases} \partial_t v + [-\nu \Delta v + (v \cdot \nabla) v + \nabla p] \; dt = G \; dw \\ \nabla \cdot v = 0 \end{cases} \tag{1}$$

where $v = \vec{v}(t, \vec{\xi}), p = p(t, \vec{\xi})$ with $t \geq 0, \vec{\xi} \in D \subset \mathbb{R}^d$ for $d = 2, 3$

$+$ initial condition

$+$ boundary condition

Why to add a stochastic term?
numerical and empirical uncertainties $\rightsquigarrow$ Gaussian model

Main issues for a stochastic PDE

- well posedness (for weak or strong solutions)
- analysis of the law (the solution is a stochastic process):
  - invariant measures ($\exists$ and !)
  - asymptotic behaviour (as $t \to \infty$)

⋮

The first paper on Navier-Stokes equations with time white noise forcing term is by Bensoussan-Temam (1973).
In the last 25 years a lot of results have been obtained for the stochastic Navier-Stokes equations.

For $d = 2$ there are results about existence and uniqueness of solutions (good domain, good spatial regularity $G$ of the noise) and properties about invariant measures: existence and uniqueness (ergodicity, ....).

What can we say about the invariant measure (when it exists and is unique)?

Gallavotti rose a question[1]:

can the law of the solution process to the stochastic Navier-Stokes equations be expressed in terms of the law of the solution process to the stochastic Stokes (**linear**) equations by means of Girsanov transform?

Girsanov transform is a technique to investigate the relationship between two SDE/SPDE's with the same noise but a different drift term.

$$\begin{cases} \partial_t v + [-\nu \Delta v + (v \cdot \nabla) v + \nabla p]\ dt = G\ dw \\ \nabla \cdot v = 0 \end{cases}$$

$$\begin{cases} \partial_t v + [-\nu \Delta v \qquad\qquad + \nabla p]\ dt = G\ dw \\ \nabla \cdot v = 0 \end{cases}$$

[1] *Ipotesi per una introduzione alla Meccanica dei fluidi* (1996) - Alta turbolenza. See also *Foundations of Fluid Dynamics* (2005) Springer, Berlin

Gallavotti rose a question[1]:

can the law of the solution process to the stochastic Navier-Stokes equations be expressed in terms of the law of the solution process to the stochastic Stokes (**linear**) equations by means of Girsanov transform?

Girsanov transform is a technique to investigate the relationship between two SDE/SPDE's with the same noise but a different drift term.

$$\begin{cases} \partial_t v + [-\nu \Delta v + (v \cdot \nabla) v + \nabla p] \ dt = G \ dw \\ \nabla \cdot v = 0 \end{cases}$$

$$\begin{cases} \partial_t v + [-\nu \Delta v \qquad\qquad\qquad + \nabla p] \ dt = G \ dw \\ \nabla \cdot v = 0 \end{cases}$$

[1] *Ipotesi per una introduzione alla Meccanica dei fluidi* (1996) - Alta turbolenza. See also *Foundations of Fluid Dynamics* (2005) Springer, Berlin

Gallavotti provided some formal expressions in the stationary case.
We investigate mathematically this question assuming

- additive noise
- "full" noise
- good spatial regularity of the noise ($G$)
- on the torus
- hyperviscosity: $-\Delta \rightsquigarrow (-\Delta)^{1+c}$ for some $c > 0$

Consequences of Girsanov theorem:
the law of the process solving the stochastic nonlinear eq is equivalent to the law the process solving the stochastic linear eq (with the same noise) and the Radon-Nikodym derivative is known.

Gallavotti provided some formal expressions in the stationary case.
We investigate mathematically this question assuming

- additive noise
- "full" noise
- good spatial regularity of the noise ($G$)
- on the torus
- hyperviscosity: $-\Delta \rightsquigarrow (-\Delta)^{1+c}$ for some $c > 0$

Consequences of Girsanov theorem:
the law of the process solving the stochastic nonlinear eq is equivalent to the law the process solving the stochastic linear eq (with the same noise) and the Radon-Nikodym derivative is known.

Some remarks for the stochastic Navier-Stokes equations ($c = 0$) will be given at the end.

Gallavotti provided some formal expressions in the stationary case.
We investigate mathematically this question assuming

- additive noise
- "full" noise
- good spatial regularity of the noise ($G$)
- on the torus
- hyperviscosity: $-\Delta \rightsquigarrow (-\Delta)^{1+c}$ for some $c > 0$

Consequences of Girsanov theorem:
the law of the process solving the stochastic nonlinear eq is equivalent to the law the process solving the stochastic linear eq (with the same noise) and the Radon-Nikodym derivative is known.

Some remarks for the stochastic Navier-Stokes equations ($c = 0$) will be given at the end.

Let us modify the dissipation

$$\begin{cases} \partial_t v + [\nu(-\Delta)^{1+c} v + (v \cdot \nabla) v + \nabla p] \; dt = G \; dw \\ \nabla \cdot v = 0 \end{cases} \tag{2}$$

This appears in some models (e.g. in simulations)

For simplicity we consider an additive noise and we work on the torus $\mathbb{R}^d/(2\pi\mathbb{Z})^d$.

In abstract form (setting $\nu = 1$)

$$dv + [A^{1+c} v + B(v, v)] \; dt = G dw \tag{3}$$

where

$A = -\Delta$

$B(u, v) = P[(u \cdot \nabla) v]$

$H^s$ is the closure in $[H^s([0, 2\pi]^d)]^d$ of the space of smooth divergence free periodic vectors

$w$ is a cylindrical Wiener process in $H^0$

$G$ is a linear operator in $H^0$

Let us modify the dissipation

$$\begin{cases} \partial_t v + [\nu(-\Delta)^{1+c} v + (v\cdot\nabla)v + \nabla p]\ dt = G\ dw \\ \nabla\cdot v = 0 \end{cases} \tag{2}$$

This appears in some models (e.g. in simulations)

For simplicity we consider an additive noise and we work on the torus $\mathbb{R}^d/(2\pi\mathbb{Z})^d$.

In abstract form (setting $\nu = 1$)

$$dv + [A^{1+c} v + B(v,v)]\ dt = Gdw \tag{3}$$

where

$A = -\Delta$

$B(u,v) = P[(u\cdot\nabla)v]$

$H^s$ is the closure in $[H^s([0,2\pi]^d)]^d$ of the space of smooth divergence free periodic vectors

$w$ is a cylindrical Wiener process in $H^0$

$G$ is a linear operator in $H^0$

- $d = 2$: global existence and uniqueness for $c \geq 0$ (no need to introduce $c > 0$ as in the deterministic setting)
- $d = 3$: global existence and uniqueness for $c \geq \frac{1}{4}$ (as in the deterministic setting, see Lions '54; for $c > \frac{1}{4}$ in the stochasting setting see Mattingly-Sinai '99)

More precisely, taking $G = A^{-\gamma}$

**Theorem ($d = 3$)**

*Let $c \geq \frac{1}{4}$ and $\gamma > \frac{3}{4}$. Given any $v_0 \in H^1$ there exists a unique process $v$ solving the hyperviscous stochastic Navier-Stokes equation (3) and such that*

$$v \in C([0, T]; H^1) \cap L^{2(1+\frac{1}{c})}(0, T; H^{1+c}) \qquad \mathbb{P} - a.s.$$

*This process is progressively measurable and Markov process in $H^1$.*

The proof of this results is similar to that of the deterministic equation (actually we work pathwise).

How to characterise the law of the solution process on the Borelian subsets of $C([0,T];H^1)$?
By a modification of a result of Liptser and Shiryaev
if the eqs

$$dv + [A^{1+c}v + B(v,v)]\ dt = Gdw$$

$$dz + A^{1+c}z\ dt \qquad\qquad = Gdw$$

are well posed (here both have strong solution in the probabilistic sense),

$$\mathbb{P}\{\int_0^T \|G^{-1}B(z(t),z(t))\|^2_{L^2}dt < \infty\} = 1$$

is enough to get that the law $\mathcal{L}_v$ of the process solving the non-linear eq is absolutely continuous w.r. to the law of the process $\mathcal{L}_z$ solving the linear eq (also the viceversa holds).
This is much weaker than the Novikov condition

$$\mathbb{E}[\exp \frac{1}{2}\int_0^T \|G^{-1}B(z(s),z(s))\|^2_{L^2}ds] < \infty$$

Let us analyze the laws of the solution processes by means of Girsanov transform (change of drift)

$$dv + [A^{1+c}v + B(v,v)]\, dt = A^{-\gamma} dw; \qquad v(0) = x \qquad (4)$$

$$dz + A^{1+c}z\, dt = A^{-\gamma} dw; \qquad z(0) = x \qquad (5)$$

## Theorem (equivalence of laws)

*Consider* $\gamma = \frac{1}{2}$ *and* $c > \frac{d}{2}$.
*Given* $x \in H^2$, *on any finite time interval* $[0,T]$ *there exists a unique weak solution of equation* (4). *Its law* $\mathcal{L}_{NS}$ *is equivalent to the law* $\mathcal{L}_S$ *of equation* (5), *as measures on the Borel subsets of* $C([0,T];H^2)$.
*Moreover*

$$\frac{d\mathcal{L}_{NS}}{d\mathcal{L}_S}(z) = \mathbb{E}\left[ e^{-\int_0^T \langle A^{\frac12}B(z(s),z(s)),dw(s)\rangle - \frac12 \int_0^T \|A^{\frac12}B(z(s),z(s))\|^2_{L^2} ds} \Big| \sigma_T(z) \right]$$

$\mathbb{P}$-*a.s.*

The proof is classical; one works on the linear equation

$$dz + A^{1+c}z\ dt = A^{-\gamma}dw$$

and then on the auxiliary equation for $u = v - z$

$$\frac{du}{dt} + A^{1+c}u + B(u+z, u+z) = 0$$

pathwise.
Notice that the bigger is $\gamma$ the smoother is the solution but the more difficult is to estimate

$$\int_0^T \|A^\gamma B(z(s), z(s))\|^2_{L^2} ds$$

Hence one has to find $\gamma$ big enough but not too much.
This is the issue to take into account in the proof.

To say that the laws $\mathcal{L}_S$ and $\mathcal{L}_{NS}$ are equivalent implies that what holds $\mathbb{P}$-a.s. for one law is valid for the other too.
Moreover, for the asymptotic behaviour and invariant measures we have the following result.
Let $P(t, x, \cdot)$ denote the law of $v(t)$ with $v(0) = x$ solving the nonlinear eq (4), and $\mu_l$ be the unique invariant measure of the linear eq (5). Then

**Proposition**

*Let $\gamma = \frac{1}{2}$ and $c > \frac{d}{2}$.*
*For equation (4) we have that $P(t, x, \cdot) \sim \mu_l$ for any $t > 0, x \in H^2$, where $\mu_l = \mathcal{N}(0, \frac{1}{2\nu} A^{-2-c})$ is the unique invariant measure for (5). In particular, the solution process u is Feller and irreducible in $H^2$ at any time $t > 0$; hence there exists at most one invariant measure for (4), which is equivalent to $\mu_l$.*

(for $d = 2$: same result as Mattingly and Suidan, J. Stat. Physics (2005) *The small scales of the Stochastic Navier-Stokes Equations under rough forcing*)

# Remark.

So there is equivalence of the laws for the linear and the nonlinear eqs if

- $c > 1$ (i.e. $1 + c > 2$) for $d = 2$
- $c > \frac{3}{2}$ (i.e. $1 + c > \frac{5}{2}$) for $d = 3$

This is not surprising since $c$ has to be taken big enough.
It implies that it is enough to look at the easier linear eq and then characterise the nonlinear eq (its solution) by means of the Radon-Nikodym derivative.

Is this a good or bad result?

Are the hyperviscous eqs good model for fluids?

What can be said for smaller values of $c$?

# Remark.

So there is equivalence of the laws for the linear and the nonlinear eqs if

- $c > 1$ (i.e. $1 + c > 2$) for $d = 2$
- $c > \frac{3}{2}$ (i.e. $1 + c > \frac{5}{2}$) for $d = 3$

This is not surprising since $c$ has to be taken big enough.
It implies that it is enough to look at the easier linear eq and then characterise the nonlinear eq (its solution) by means of the Radon-Nikodym derivative.

Is this a good or bad result?

Are the hyperviscous eqs good model for fluids?

What can be said for smaller values of $c$?

In a paper by Frisch, Kurien, Pandit, Pauls, Ray, Wirth, Zhu (PRL 101 (2008), "Hyperviscosity, Galerkin Truncation, and Bottlenecks in Turbulence") they write

Hyperviscosity is frequently used in turbulence modeling to avoid wasting numerical resolution by reducing the range of scales over which dissipation is effective. ... Artifacts arising from models with $c > 0$ are discussed. ...

We finally deal with the case of moderately large $c$ of the sort used in many simulations. How safe is this procedure and what kind of artifacts can we expect? Using large values of $c$ in simulations to avoid wasting resolution is hardly advocated by anybody, but we now understand what goes wrong: a huge thermalized bottle-neck will develop at high wave numbers, whose action on smaller wave numbers is an ordinary $c = 0$ dissipation with an eddy viscosity much larger than what would be permissible in a normal $c = 0$ simulation. When $c$ is chosen just a bit larger than 0 (e.g., $c = 1$ which is standard in oceanography) the advantage of widening the inertial range may be offset by artifacts at bottleneck scales; indeed, even an incomplete thermalization will bring the statistical properties of such scales closer to Gaussian, thereby reducing the rather strong intermittency which would otherwise be expected.

We analyse the case $d = 3$ by working on the hyperviscous Navier-Stokes eqs in the vorticity form.
The aim is to find an easier reference equation....

The 3D hyperviscous Navier-Stokes equations in **vorticity form** (define the vorticity as $\omega = \nabla \times \vec{v}$)

$$\begin{cases} d\vec{\omega} + \Big( (-\Delta)^{1+c}\vec{\omega} + \underbrace{(\vec{v}\cdot\nabla)\vec{\omega}}_{transport} - \underbrace{(\vec{\omega}\cdot\nabla)\vec{v}}_{stretching} \Big)\, dt = (-\Delta)^{-b} d\vec{w} \\ \nabla \cdot \vec{v} = 0 \\ \vec{\omega} = \nabla \times \vec{v} \end{cases} \tag{6}$$

are well posed for $c \geq \frac{1}{4}$

The 3D hyperviscous vorticity transport equations

$$\begin{cases} d\vec{\eta} + \big( (-\Delta)^{1+c}\vec{\eta} + P[(\vec{v}\cdot\nabla)\vec{\eta}] \big)\, dt = (-\Delta)^{-b} d\vec{w} \\ \nabla \cdot \vec{v} = 0 \\ \vec{\eta} = \nabla \times \vec{v} \end{cases} \tag{7}$$

are well posed for $c \geq 0$

The 3D hyperviscous Navier-Stokes equations in **vorticity form** (define the vorticity as $\omega = \nabla \times \vec{v}$)

$$\begin{cases} d\vec{\omega} + \Big( (-\Delta)^{1+c}\vec{\omega} + \underbrace{(\vec{v}\cdot\nabla)\vec{\omega}}_{transport} - \underbrace{(\vec{\omega}\cdot\nabla)\vec{v}}_{stretching} \Big)\, dt = (-\Delta)^{-b} d\vec{w} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\omega} = \nabla\times\vec{v} \end{cases} \tag{6}$$

are well posed for $c \geq \frac{1}{4}$

The 3D hyperviscous vorticity transport equations

$$\begin{cases} d\vec{\eta} + \big((-\Delta)^{1+c}\vec{\eta} + P[(\vec{v}\cdot\nabla)\vec{\eta}]\big)\ dt = (-\Delta)^{-b} d\vec{w} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\eta} = \nabla\times\vec{v} \end{cases} \tag{7}$$

are well posed for $c \geq 0$

# The vorticity transport equation

Pathwise analysis of the equation.

**Theorem**

*i) Assume that*

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{1}{2} \end{cases}$$

*Then, for any $\vec{\eta}(0) \in L_2$ there exists a unique process $\vec{\eta}$ solving (7) such that*

$$\vec{\eta} \in C([0,T];L_2) \cap L^2(0,T;L_6)$$

$\mathbb{P}$*-a.s.. Moreover there is continuous dependence on the initial data: given two initial data $\vec{\eta}(0), \vec{\eta}_\star(0) \in L_2$ we have*

$$\|\vec{\eta}(0) - \vec{\eta}_\star(0)\|_{L_2} \to 0 \Longrightarrow \|\vec{\eta} - \vec{\eta}_\star\|_{C([0,T];L_2)} \to 0$$

## Theorem (continuation)

*ii) Assume that*

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{3}{2} \end{cases}$$

*Then, for any $\vec{\eta}(0) \in H^1$ the solution given in i) enjoys also*

$$\vec{\eta} \in C([0,T]; H^1) \qquad \mathbb{P} - a.s.$$

*iii) Assume that*

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{5}{2} \end{cases}$$

*Then, for any $\vec{\eta}(0) \in H^2$ the solution given in i) enjoys also*

$$\vec{\eta} \in C([0,T]; H^2) \qquad \mathbb{P} - a.s.$$

# The NS equation in vorticity form

### Theorem

*i) Assume that*

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \end{cases}$$

*Then, for any* $\vec{\omega}(0) \in L_2$ *there exists a unique process* $\vec{\omega}$ *solving* (6) *such that*

$$\vec{\omega} \in C([0, T]; L_2) \cap L^2(0, T; L_6)$$

$\mathbb{P}$*-a.s.*

*Moreover there is continuous dependence on the initial data: given two initial data* $\vec{\omega}(0), \vec{\omega}_\star(0) \in L_2$ *we have*

$$\|\vec{\omega}(0) - \vec{\omega}_\star(0)\|_{L_2} \to 0 \Longrightarrow \|\vec{\omega} - \vec{\omega}_\star\|_{C([0,T];L_2)} \to 0$$

## Theorem (continuation)

*ii) Assume that*

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \\ 2b + c > \frac{3}{2} \end{cases}$$

*Then, for any $\vec{\omega}(0) \in H^1$ the solution given in i) enjoys also*

$$\vec{\omega} \in C([0, T]; H^1) \qquad \mathbb{P} - a.s.$$

*iii) Assume that*

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \\ 2b + c > \frac{5}{2} \end{cases}$$

*Then, for any $\vec{\omega}(0) \in H^2$ the solution given in i) enjoys also*

$$\vec{\omega} \in C([0, T]; H^2) \qquad \mathbb{P} - a.s.$$

If the two eqs are well posed, we investigate Girsanov transform.
Let $\mathcal{T}$ denote the inverse of the curl (Biot-Savart law).

$$d\vec{\omega} + \Big((-\Delta)^{1+c}\vec{\omega} + P[(\mathcal{T}\vec{\omega}\cdot\nabla)\vec{\omega}] - P[(\vec{\omega}\cdot\nabla)\mathcal{T}\vec{\omega}]\ \Big)\ dt = (-\Delta)^{-b} d\vec{w}$$

$$d\vec{\eta} + \big((-\Delta)^{1+c}\vec{\eta} + P[(\mathcal{T}\vec{\eta}\cdot\nabla)\vec{\eta}]\big)\ dt \qquad = (-\Delta)^{-b} d\vec{w}$$

### Theorem (equivalence of laws)

*Let*

$$\begin{cases} c > \frac{1}{2} \\ b = 1 \end{cases}$$

*If* $\vec{\eta}(0) = \vec{\omega}(0) \in H^2$, *then the laws* $\mathcal{L}_{\vec{\omega}}$ *and* $\mathcal{L}_{\vec{\eta}}$ *are equivalent and*

$$\frac{d\mathcal{L}_{\vec{\omega}}}{d\mathcal{L}_{\vec{\eta}}}(\vec{\eta}) = \mathbb{E}\left[e^{\int_0^T \langle(-\Delta)^b P[(\vec{\eta}(t)\cdot\nabla)\mathcal{T}\vec{\eta}(t)], d\vec{w}(s)\rangle - \frac{1}{2}\int_0^T \|(-\Delta)^b P[(\vec{\eta}(t)\cdot\nabla)\mathcal{T}\vec{\eta}(t)]\|^2_{L_2} ds}\Big|\sigma \tag{8}$$

$\mathbb{P}$-*a.s.*

We have got a lower bound for $c$ ($c > \frac{1}{2}$ instead of $c > \frac{3}{2}$ in $d = 3$) but we paid that the law of the vorticity transport equation has no explicit expression.

CONCLUSION

From the mathematical point of view we have given a quantitative information (how large $c$ must be) in order to get the Girsanov representation of the law.
This makes easier the analysis of the stochastic hyperviscous Navier-Stokes eq.

# Same invariant measure

Can one expect that the Stokes and the Navier-Stokes eqs share the same invariant measure?

Techniques from statistical mechanics:
keeping in mind that the Euler eq has two quadratic invariants: the energy (for $d = 2, 3$) and the enstrophy (for $d = 2$)

Gibbs measure of the energy

$$\mu_0(dv) ='' \frac{1}{Z} e^{-\frac{1}{2}\|v\|^2_{L^2}} dv''$$

is formally invariant for the Navier-Stokes eq with very irregular (in space) noise

$$dv + [Av + B(v,v)]\ dt = A^{\frac{1}{2}} dw$$

(for $d = 2, 3$). Notice that supp $\mu_0 = \cap_{s<-\frac{d}{2}} H^s$.

This comes from the fact that $\mu_0$ is an invariant measure for the Stokes eq

$$dv + Av\ dt = A^{\frac{1}{2}} dw$$

and (formally or infinitesimally) invariant for the Euler eq

$$\frac{dv}{dt} + B(v,v) = 0$$

The analysis of these eqs is difficult.
Something has been done by Gubinelli-Jara (2013) when $d = 2$ and

$$dv + [A^{1+\epsilon} v + B(v,v)]\ dt = A^{\frac{1}{2}+\frac{\epsilon}{2}} dw$$

For $d = 2$ the Gibbs measure of the enstrophy[2]

$$\mu_1(dv) =" \frac{1}{Z} e^{-\frac{1}{2}\| \text{ curl } v\|^2_{L^2}} dv"$$

is invariant for the Navier-Stokes eq with less irregular (in space) noise[3]

$$dv + [Av + B(v, v)]\ dt = dw$$

Now supp $\mu_1 = \cap_{s<0} H^s$, $\mu_1$ is an invariant measure for the Stokes eq

$$dv + Av\ dt = dw$$

and (formally or infinitesimally) invariant for the Euler eq

$$\frac{dv}{dt} + B(v, v) = 0$$

Here the conservation of the enstrophy for Euler eq comes in and this result requires to work on the torus (same b.c. for Euler and for Navier-Stokes).

[2] see Albeverio, de Faria, Høegh-Khron (1979)

[3] see Albeverio, Cruzeiro (1990), Da Prato, Debussche (2003), Albeverio, F (2004), Zhu, Zhu (2017)

BIBLIOGRAPHY


- B. Ferrario: *Characterization of the law for 3D stochastic hyperviscous fluids*, Electron. J. Probab. 21 (2016), paper no. 26, 22 pp.
- B. Ferrario: *Uniqueness and absolute continuity for semilinear SPDE's*, (2013), 85-94. Proceedings of "Seminar on Stochastic Analysis, Random Fields and Applications-VII", Ascona 2011. Birkhuser. Series: Progress in Probability, Vol. 67
- B. Ferrario: *A note on a result of Liptser-Shiryaev*, Stochastic Analysis and Applications 30 no.6 (2012), 1019-1040
- B. Ferrario: *Well posedness of a stochastic hyperviscosity-regularized 3D Navier-Stokes equation*, Proceedings of the Conference "SPDE's and Applications - VIII" (Levico, Jan. 6-12, 2008), Quaderni di Matematica 25 - Seconda Università di Napoli (2010), 127-138
- B. Ferrario: *Absolute continuity of laws for semilinear stochastic equations with additive noise*, Communications on Stochastic Analysis 2 no.2 (2008), 209-227. Erratum 5 no.2 (2011), 431-432

Thank you!