Existence of vortex sheets for 2D stochastic Euler equations

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Motivation: in the deterministic case, we have existence of solutions to 2D Euler equations with singular vorticity, in particular vortex sheets (initial vorticity concentrated on a line).
Question: what for *stochastic* 2D Euler equations?
Main result: existence of a martingale solution to the 2D stochastic Euler equations with transport noise:

$$\partial_t \xi + u \cdot \nabla \xi + \sum_k \sigma_k \cdot \nabla \xi \circ \dot{W}^k = 0$$

$$\xi = \text{const} + \text{curl} u$$

(σ_k given vector fields, W^k independent Brownian motions) when the vorticity is a non-negative measure and in H^{-1}. This includes vortex sheets.

2D Euler equations (on $\mathbb{R}^2$ or $\mathbb{T}^2$)

$$\partial_t u + (u \cdot \nabla) u = -\nabla p$$
$$\mathrm{div} u = 0$$

In vorticity form: $\xi = \text{const} + \text{curl u}$ (scalar valued):

$$\partial_t \xi + u \cdot \nabla \xi = 0,$$
$$u = K * \xi$$

where $K = \nabla^{\perp} G$ and G is the Green function of the Laplacian. Note $K(x) \approx \frac{x^{\perp}}{|x|^2}$. Solutions in distributional form.

Results:

- Vorticity ξ in L^∞_x: existence and uniqueness (Yudovich 63, Marchioro-Pulvirenti 94).
- Vorticity ξ in L^p, $p > 1$: existence (DiPerna-Majda 87).
- Vorticity ξ in $\mathcal{M}_+ \cap H^{-1}$: existence (Delort 91).

Comparison with Onsager dissipative solutions:

- Vorticity in L^p: solutions u are more regular ($W^{1,p}$) than Hölder. From DiPerna-Lions and Ambrosio theory: existence of a Lagrangian flow, renormalization expected for bounded u; actually renormalization and energy conservation hold for $p > 3/2$ and also below under some restrictions.
- Vorticity in $\mathcal{M}_+ \cap H^{-1}$: not known.

Now we introduce noise. Noise interpretation:

- uncertainties
- can give rise to turbulent phenomena
- can improve well-posedness theory w.r.t. the deterministic case (regularization by noise)

Here: Extension of Delort existence result to stochastic case.

Stochastic 2D Euler equations with transport noise (on the vorticity):

$$\partial_t \xi + u \cdot \nabla \xi + \sum_k \sigma_k \cdot \nabla \xi \circ \dot{W}^k = 0$$
$$u = K * \xi$$

Here σ_k are given divergence-free vector fields, assumed regular, and W^k are independent real Brownian motions, $\circ$ denotes Stratonovich integration.
Solution: $\xi : [0, T] \times \mathbb{T}^2 \times \Omega \to \mathbb{R}$ random scalar field.

- Noise interpretation: Calling formally $\zeta(t, x) = \sum_k \sigma_k(x) \dot{W}^k$, ζ is a random field, Gaussian, decorrelated in time, correlated and smooth in space.

Features of transport noise:

1) Solution follows stochastic characteristics of the fluid (formally):

$$\xi_t(X_t) = \xi_0$$
$$dX_t = u(t, X_t)dt + \sigma_k(X_t) \circ dW_t$$

2) Stochastic Constantin-Iyer formula (proved by Flandoli-Luo for the 3D case)

$$u_t = \Pi[(\nabla X_t^{-1})^T u_0(X_t^{-1})]$$

3) As consequence of transport property: enstrophy and any L^p norm are preserved.

In velocity form:

$$\partial_t u + (u \cdot \nabla) u + [(\sigma_k \cdot \nabla) u + (\nabla \sigma_k)^T u] \circ \dot{W}^k = -\nabla p$$
$$\mathrm{div} u = 0$$

Note: zero order term, energy (that is L^2 norm of u) is not preserved.

Results:

- Vorticity in L^∞: existence and uniqueness, in pathwise sense (Brzezniak-Flandoli-M. 16). Idea of proof (in the line of Marchioro-Pulvirenti 94): prove uniqueness of stochastic characteristics (stochastic flows), then prove renormalization-type property.
- Vorticity in L^2, transport noise in the velocity: existence of a martingale solution (Yokoyama 14, with a similar technique to the one here).

Other results:

- Crisan-Flandoli-Holm 17: local existence and uniqueness in 3D.
- Flandoli-Gubinelli-Priola 11: regularization by noise for vorticity concentrated in a finite number of points.

A few examples of other noises:

- Mikulevicius–Valiukevicius 00: local existence of smooth solutions under 2D $(+\dot{W}^1 + \dot{W}^2)$ noise.
- Bessaih–Flandoli 99, Bessaih 99: existence of a martingale solution under affine multiplicative noise $+\sum_k \sigma_k(x)u\dot{W}^k$.
- Brzezniak–Peszat 01: existence of a martingale solution in L^2 under multiplicative noise $+G(u)dW$, $G(u)$ in $W^{1,p}$ (roughly speaking.
- Glatt-Holtz–Vicol 14: existence of smooth solutions under additive noise $(\sum_k \sigma_k(x)\dot{W}^k)$ and linear multiplicative $(\alpha u\dot{W}^k)$ noise.

Stochastic 2D Euler equations on the torus $\mathbb{T}^2$:

$$\partial_t \xi + u \cdot \nabla \xi + \sum_k \sigma_k \cdot \nabla \xi \circ \dot{W}^k = 0$$
$$u = K * \xi$$

Assumptions on σ_k:

- divergence-free
- regular: $\sum_k \|\sigma_k\|_{C^2}^2 < \infty$
- the covariance matrix $C(x,y) = \sum_k \sigma_k(x)\sigma_k(y)^T$ is locally translation-invariant and even (OK if the noise is "locally isotropic")

Theorem (Brzezniak-M.)

Assume σ_k as above. Let ξ_0 be in $\mathcal{M}_+ \cap H^{-1}$. Then there exists a weak (in the probabilistic sense) solution ξ with a.s. values in $C_t(\mathcal{M}_+, w*) \cap L^2_t(H^{-1})$.

Remarks:

- We chose the torus as the simplest case, we expect the result to hold also on $\mathbb{R}^2$.
- The assumptions on the structure of the covariance matrix associated with σ_k may be relaxed, they are put to simplify Itô-Stratonovich corrections.

How to make sense of $u \cdot \nabla \xi$ for ξ measure? u is in general not bounded.
Poupaud 02 trick: since K is odd, we can write formally

$$\begin{aligned}\int u(x)\xi(x)\nabla\varphi(x)dx &= \int\int \xi(x)\xi(y)K(x-y)\nabla\varphi(x)dx \\ &= \frac{1}{2}\int\int \xi(x)\xi(y)F_\varphi(x,y)dx\end{aligned}$$

where

$$F_\varphi(x,y) = K(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))$$

Recall $K(x-y) \approx \frac{(x-y)^\perp}{|x-y|^2}$. Therefore, for φ in C^2, F_φ is regular outside the diagonal $\{x=y\}$ and bounded everywhere.

Hence, for ξ general measure, we define $u \cdot \nabla \xi$ as

$$\langle u \cdot \nabla \xi, \varphi \rangle := \int \int \xi(x)\xi(y) F_{\varphi}(x,y) dx dy$$

Note that

$$|\langle u \cdot \nabla \xi, \varphi \rangle| \leq C \|\xi\|_{\mathcal{M}}^2 \|\varphi\|_{C^2} \leq C \|\xi\|_{\mathcal{M}}^2 \|\varphi\|_{H^4}$$

that is

$$\|u \cdot \nabla \xi\|_{H^{-4}} \leq C \|\xi\|_{\mathcal{M}}^2$$

Strategy: 1) tightness 2) equation for any limiting object
1) Tightness: approximation by regular solutions, uniform $L^\infty_{t,\omega}(\mathcal{M})$, $L^2_{t,\omega}(H^{-1})$ and $L^2_\omega(C^\alpha_t(H^{-4}))$ bounds, via transport structure, Poupaud trick, stochastic C^α bounds.
2) Equation for the limiting objects: a.s. convergence (Skorohod-Jakubowski theorem), convergence of nonlinear term by Poupaud trick.

Take ξ^ϵ bounded solutions to stochastic Euler equations but with regular initial data ξ^ϵ approximating ξ.
1) Uniform $L^\infty_{t,\omega}(\mathcal{M})$ bound: transport structure implies mass conservation:

$$\partial_t \int \xi^\epsilon dx = -\int u^\epsilon \cdot \nabla \xi^\epsilon dx - \int \sigma_k \cdot \nabla \xi^\epsilon dx \circ \dot{W}^k = 0$$

2) Uniform $L^\infty_t(L^2_\omega(H^{-1}))$ bound: equivalent to uniform $L^\infty_t(L^2_\omega(L^2))$ bound on u^ϵ (energy bound):
Equation for u^ϵ:

$$\partial_t u^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon + (\sigma_k \cdot \nabla u^\epsilon) \circ \dot{W}^k + (\nabla \sigma_k)^T u^\epsilon \circ \dot{W}^k = -\nabla p^\epsilon$$

Assumptions on σ_k imply that $(\nabla \sigma_k)^T u^\epsilon \circ \dot{W}^k = (\nabla \sigma_k)^T u^\epsilon \dot{W}^k$.
Get equation for $|u^\epsilon|^2$ and integrate in x and ω:

$$\partial_t \mathbb{E} \int |u^\epsilon|^2 dx = \mathbb{E} \int |(\nabla \sigma_k)^T u^\epsilon|^2 dx \le C\mathbb{E} \int |u^\epsilon|^2 dx$$

Conclusion by Gronwall lemma.

3) Uniform $L^2_\omega(C^\alpha_t(H^{-4}))$ bound, $\alpha < 1/2$: use Poupaud trick and stochastic calculus:

$$u_t - u_s = -\int_s^t u \cdot \nabla \xi dr - \int_s^t \sigma_k \cdot \nabla \xi dW^k + \frac{1}{2}\int_s^t \mathrm{tr}[C(0)D^2\xi]dr$$

Nonlinear term: $\|u \cdot \nabla \xi\|_{H^{-4}} \leq C\|\xi\|^2_{\mathcal{M}}$, hence Lipschitz in time.
Stochastic term: $\|\sigma_k \cdot \nabla \xi\|_{H^{-4}} \leq C\|\sigma_k\|_C\|\xi\|_{\mathcal{M}}$, hence $1/2-$ Hölder in time (stochastic integration in the Hilbert space H^{-4}).

Tightness in the space $\chi = C_t(\mathcal{M}_M, w*) \cap (L^2_t(H^{-1}_x), w)$:

- The set $A = \{\mu \in \chi \mid \|\mu\|_{L^2_t(H^{-1}_x, w)} + \|\mu\|_{C^\alpha_t(H^{-4}_x)} \leq a\}$ is compact.
- Uniform bounds before: $\mathrm{Law}(\xi^\epsilon)(A^c) < \delta$ for small ϵ.

Let ξ^{ϵ_n} be a subsequence such that (W, ξ^n) converges in law.
Skorokhod-Jakubowski theorem (Jakubowski 97 for r.v. with values in topological spaces): on an enlarged probability space, there exist r.v. $(\tilde{W}^{(n)}, \tilde{\xi}^n)$, copies of (W, ξ^{ϵ_n}), converging a.s. to a r.v. $(\tilde{W}, \tilde{\xi})$.
Adaptedness: take $\tilde{\mathcal{F}}_t$ as the completion of the filtration generated by $(\tilde{W}, \tilde{\xi})$, then $\tilde{W}$ is a (cylindrical) Brownian motion w.r.t. $\tilde{\mathcal{F}}_t$ and $\tilde{\xi}$ is adapted w.r.t. $\tilde{\mathcal{F}}_t$.

Limiting equation: show that each term in the equation for $\tilde{\xi}^n$ passes to the limit.

Nonlinear term

$$\int\int \tilde{\xi}(x)\tilde{\xi}(y)F_\varphi(x,y)dxdy$$

Poupaud trick:

- If $\tilde{\xi}(x)\tilde{\xi}(y)$ gives no mass to the diagonal $\{x=y\}$ and $\tilde{\xi}$ is positive, then the nonlinear term converges: indeed recall F_φ is continuous outside the diagonal.
- If $\tilde{\xi}$ is in H^{-1}, then $\tilde{\xi}$ has no atom and so $\tilde{\xi}(x)\tilde{\xi}(y)$ gives no mass to the diagonal.

Stochastic term: linear but not continuous functional of $\tilde{\xi}$.
Brzezniak-Goldys-Jegarai 13:
If the integrands were step functions (in time), the stochastic integral would be continuous w.r.t. $\tilde{\xi}$ and so convergence would hold. In the general case, approximate the integrand by step functions.

Further developments:

- Particle approximation for L^p vorticity (à la Schochet 96)
- More general noises (fractional Brownian motion, or also non-transport noises)
- σ_k irregular? (Kraichnan model for passive scalars)

Thank you!