Self-similar solutions to Burgers and Navier-Stokes equations

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Motivation

No self-similar solutions of Navier-Stokes equations blow up. Nevertheless it is of interest to see they actually behave.

Cannone-Planchon(1996) proved the existence of forward self-similar solutions for small data by using a fixed-point theorem. For recent developments, see e.g. Chae-Wolf (2018), Bradshaw-Tsai(2018).

We will try to construct asymptotic solutions, using $\nabla \times \boldsymbol{\omega}$.

One of the principal objects of theoretical research is to find the point of view from which the subject appears in the greatest simplicity. (W. Gibbs)

Plan

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5. Summary and outlook

(with Riccardo Vanon, PDRA)

0. Introduction

Theorem 2 Cannone-Planchon (1996) *Let* $u_0 \in \dot{B}^0_{3,\infty}$ *be a divergence free vector field that is homogeneous of degree -1. There exists an universal constant* $\eta > 0$ *and* $T > 0$ *such that, if*

$$\|u_0\|_{\dot{B}^0_{3,\infty}} < \eta,$$

then there exists a unique global solution $u(x,t)$ *of the Navier-Stokes equations such that*

$$u(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$$

with $U \in \dot{B}^0_{3,\infty}$ *and*

$$U = S(1)u_0 + W$$

where $W \in L^3$ *satisfies* $\|W\|_{L^3} < C(\eta)$.
($S(t)$ = heat operator)

Initial data for self-similar solution (3D)

$$\lambda \boldsymbol{u}_0(\lambda \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$$

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{\sqrt{t}} \boldsymbol{U}\left(\frac{\boldsymbol{x}}{\sqrt{t}}\right)$$

e.g. $U(\boldsymbol{x}) = \frac{1}{1+|\boldsymbol{x}|}$, $u(\boldsymbol{x}, t) = \frac{1}{|\boldsymbol{x}|+\sqrt{t}} \to \frac{1}{|\boldsymbol{x}|}$ as $t \to 0$

Construction of solutions in Besov spaces:

$\alpha > -d$ for homogeneous function of degree α

$$u \sim \frac{1}{x} \text{ in 1D}, \quad \omega \sim \frac{1}{x^2} \text{ in 2D are not included.}$$

cf. $\lambda^d \delta(\lambda \boldsymbol{x}) = \delta(\boldsymbol{x})$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nu \triangle \boldsymbol{v}, \quad \boldsymbol{v}(\boldsymbol{x}, t) = -2\nu \nabla \log \phi$$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\displaystyle\int_{\mathbb{R}^3} \frac{\boldsymbol{x} - \boldsymbol{y}}{t} \phi_0(\boldsymbol{y}) \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{4\nu t}\right) d\boldsymbol{y}}{\displaystyle\int_{\mathbb{R}^3} \phi_0(\boldsymbol{y}) \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{4\nu t}\right) d\boldsymbol{y}}$$

A few points deserve emphasis. First, this is only a particular solution to the equation. Second, even so, it is quite complicated and would still require considerable computer analysis to produce a flow field. Third, it is an irrotational solution, as mentioned.

"The Ceaseless Wind: An Introduction to the Theory of Atmospheric Motion," J.A. Dutton

2. 1D Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

static scale invariance

$$x \to \lambda x, t \to \lambda^2 t, u \to \lambda^{-1} u$$

Property 1: if $u(x,t)$ is a solution, so is $\lambda u(\lambda x, \lambda^2 t)$.

Simpler critical case with ϕ where $u = \partial_x \phi$

Property 1': if $\phi(x,t)$ is a solution, so is $\phi(\lambda x, \lambda^2 t)$.

Note velocity potential $[\phi] = [\nu] = L^2/T$

Dynamic scaling $u(x,t) = \frac{1}{\sqrt{2at}} U(\xi, \tau),$

$$\xi = \frac{x}{\sqrt{2at}}, \ \tau = \frac{1}{2a} \log(1 + 2at)$$

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} = a \frac{\partial}{\partial \xi}(\xi U) + \nu \frac{\partial^2 U}{\partial \xi^2} \quad \text{(Fokker-Planck operator)}$$

$$\frac{\partial \Phi}{\partial \tau} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \xi}\right)^2 = a\xi \frac{\partial \Phi}{\partial \xi} + \nu \frac{\partial^2 \Phi}{\partial \xi^2} \quad \text{(Ornstein-Uhlenbeck op.)}$$

quasi-invariance

Steady solution (i.e. self-similar solution)
"source solution"

$$U(\xi) = \frac{U(0)\exp\left(-\frac{a\xi^2}{2\nu}\right)}{1 - \frac{U(0)}{2\nu}\int_0^{\xi} \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta}$$

valid only for small data $U(0) < \sqrt{8a\nu/\pi}$

If $u_0 \in L^1$, for $1 \le p \le \infty$,

$$t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\| u(x,t) - \frac{1}{\sqrt{2at}} U(\xi) \right\|_p \to 0 \text{ as } t \to \infty,$$

where $\xi = \frac{x}{\sqrt{2at}}$

$$\lim_{t \to 0} \frac{1}{\sqrt{2at}} U(\cdot) = M\delta(x), \quad M \equiv \int_{\mathbb{R}^1} u_0(x)dx$$

e.g. Escobedo-Zuazua(1991), Karch(1999), Biler et al.(2001)

$$\text{cf. } U(0) = \sqrt{\frac{8a\nu}{\pi}} \tanh \frac{M}{4\nu} \approx \sqrt{\frac{a}{2\pi\nu}} M \quad (\nu \gg 1)$$

Three methods of solutions

(1) Simplest method (no linearisation)

$$\frac{U^2}{2} = a\xi U + \nu\frac{dU}{d\xi} = \nu \exp\left(-\frac{a\xi^2}{2\nu}\right)\frac{d}{d\xi}\underbrace{\left(U \exp\left(\frac{a\xi^2}{2\nu}\right)\right)}_{\equiv \tilde{U}}$$

$$2\nu \exp\left(\frac{a\xi^2}{2\nu}\right)\frac{d}{d\xi}\tilde{U} = \tilde{U}^2$$

$$\frac{d\tilde{U}}{d\eta} = \tilde{U}^2, \quad \text{by} \quad \eta \equiv \frac{1}{2\nu}\int_0^{\xi} \exp\left(-\frac{a\zeta^2}{2\nu}\right) d\zeta$$

(2) Another method (Bernoulli equation)

$$\frac{dU}{d\xi} = \frac{1}{2\nu}U^2 - \frac{a}{\nu}\xi U$$

Linearisation by $V = 1/U$

$$\frac{dV}{d\xi} = \frac{-1}{2\nu}(1 - 2a\xi V)$$

(3) Yet another (awkward) method
Observe

$$U(\xi) = \frac{U(0)\exp\left(-\frac{a\xi^2}{2\nu}\right)}{1 - \frac{U(0)}{2\nu}\int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta}$$

$$= U(0)\exp\left(-\frac{a\xi^2}{2\nu}\right) \sum_{n=1}^{\infty} \left\{\frac{U(0)}{2\nu}\int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta\right\}^n$$

$$\text{cf. } \frac{1}{1-r} = 1 + r + r^2 + \ldots, \quad |r| < 1$$

Introduce a formal series expansion

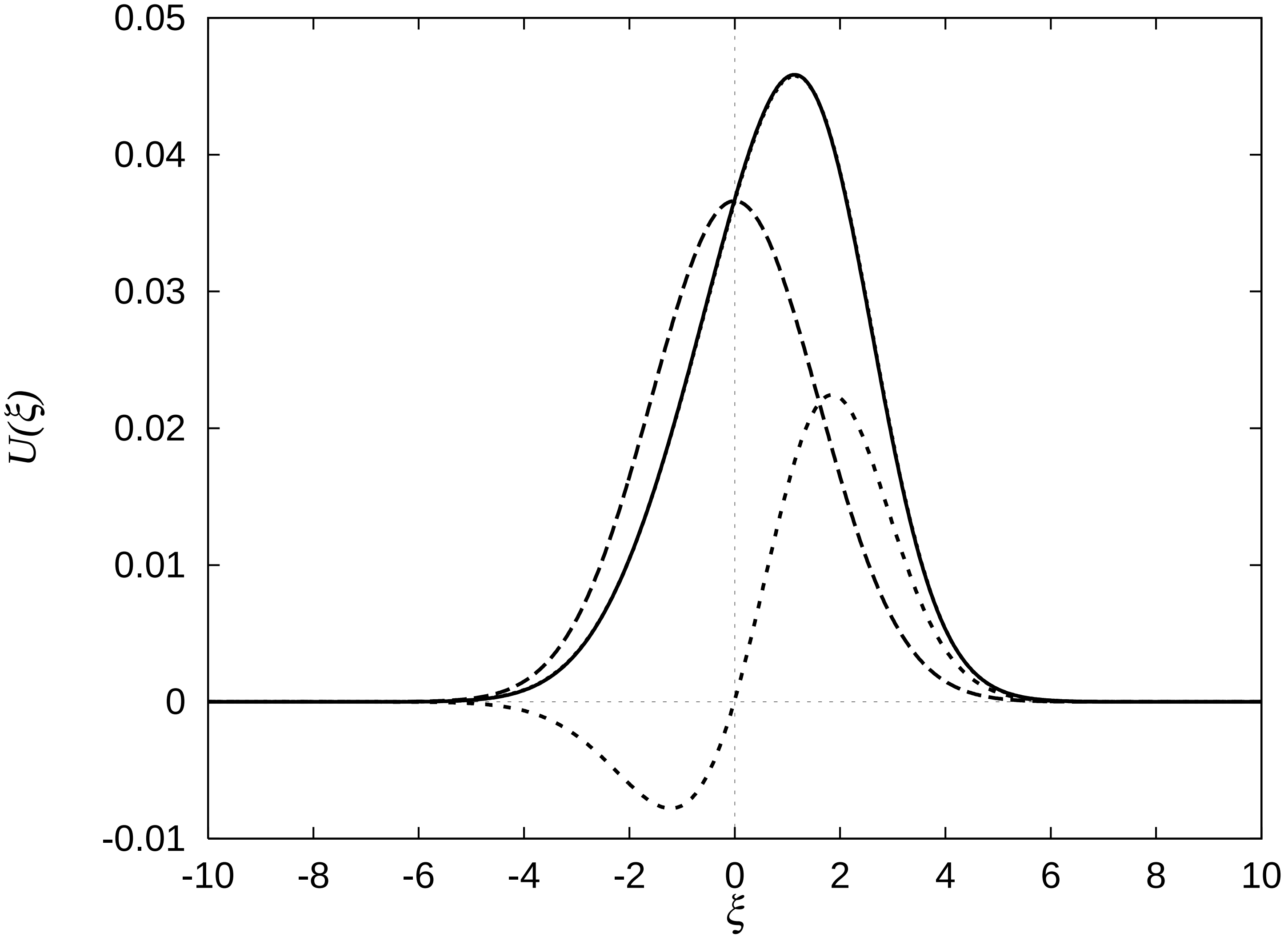
$$\tilde{U} = \epsilon U_1 + \epsilon^2 U_2 + \ldots \quad \text{in} \quad \frac{d\tilde{U}}{d\xi} = \frac{1}{2\nu} \exp\left(-\frac{a\xi^2}{2\nu}\right) \tilde{U}^2$$

$$O(\epsilon): \quad \frac{dU_1}{d\xi} = 0 \to U_1 = U(0).$$

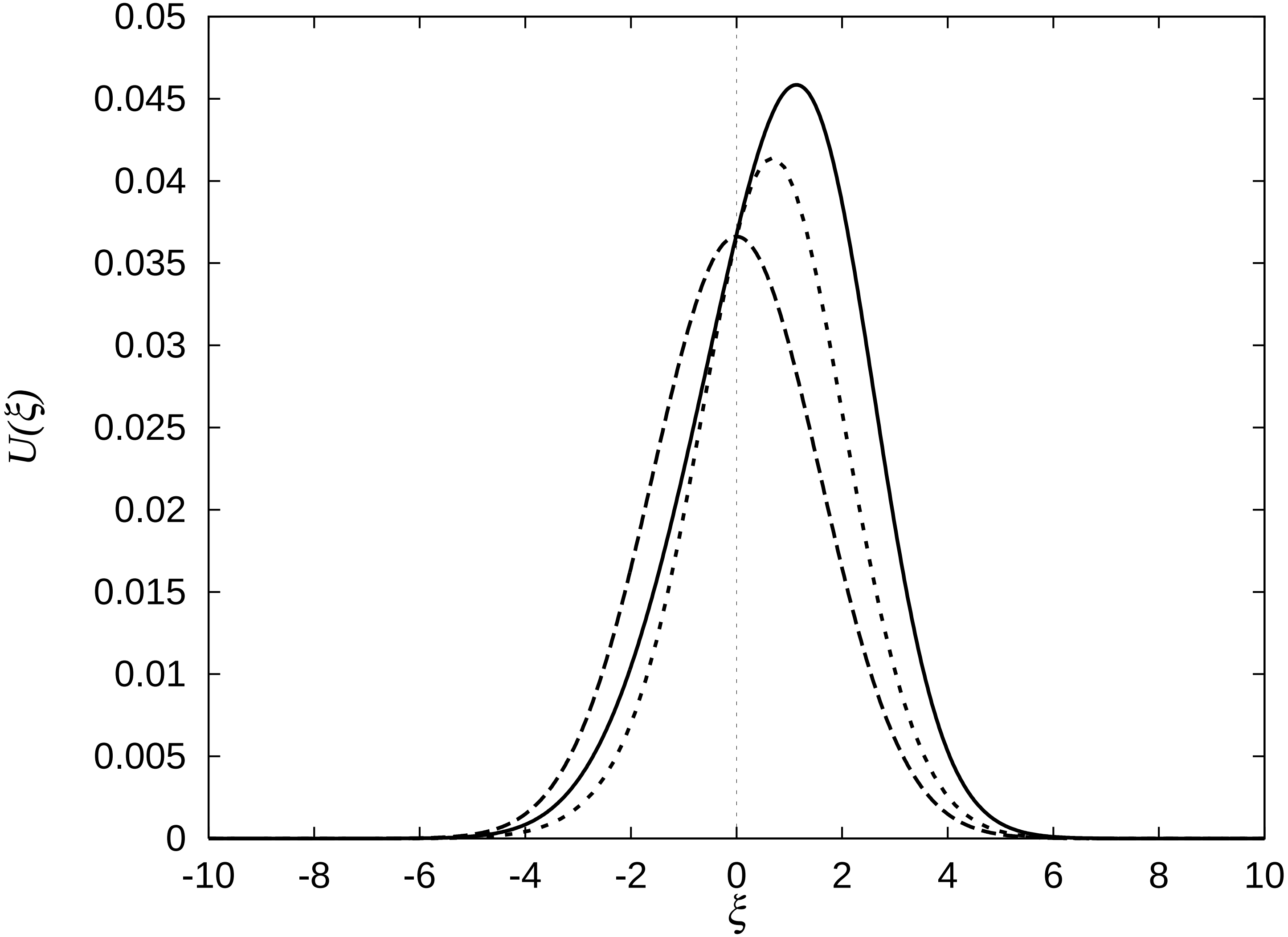
$$O(\epsilon^2): \quad \frac{dU_2}{d\xi} = \frac{1}{2\nu} \exp\left(-\frac{a\xi^2}{2\nu}\right) U_1^2$$

$$\to U_2 = \frac{U(0)^2}{2\nu} \int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta$$

$$O(\epsilon^3): \ \to U_3 = \frac{U(0)^3}{4\nu^2} \left\{ \int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta \right\}^2$$



ν=0.05,
a=0.02



0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
$U(\xi)$
-10
-8
-6
-4
-2
0
2
4
6
8
10
ξ

$$U(\xi) = \frac{U(0)\exp\left(-\frac{a\xi^2}{2\nu}\right)}{1 - \frac{U(0)}{2\nu}\int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right)d\eta}$$

$$= -2\nu\frac{\partial}{\partial\xi}\log\left\{U(0)^{-1} - \frac{1}{2\nu}\int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right)d\eta\right\}$$

step 1: find a self-similar solution

step 2: make a substitution

a self-similar heat flow

$$u(x,t) = \frac{1}{\sqrt{2at}} \frac{U(0)\exp\left(-\frac{x^2}{4\nu t}\right)}{1 - \frac{U(0)}{2\nu}\int_0^{\frac{x}{\sqrt{2at}}} \exp\left(-\frac{\eta^2}{4\nu t}\right) d\eta}$$

$\Longrightarrow$ a more general heat flow

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t}\psi_0(y)\exp\left(-\frac{(x-y)^2}{4\nu t}\right) dy}{\displaystyle\int_{-\infty}^{\infty} \psi_0(y)\exp\left(-\frac{(x-y)^2}{4\nu t}\right) dy}$$

2. Examples

Burgers equations in several dimensions

1D

$$U(\xi) = \frac{U(0)\exp\left(-\frac{a\xi^2}{2\nu}\right)}{1 - R}$$

$$R = \frac{U(0)}{2\nu}\int_0^{\xi} \exp\left(-\frac{a\xi'^2}{2\nu}\right) d\xi'$$

or

$$U(\xi) = -2\nu\frac{\partial}{\partial\xi}\log(1 - R)$$

2D

$$\frac{\partial U_1}{\partial \xi_2} = \frac{\frac{\partial U_1}{\partial \xi_2}(0)\exp\left(-\frac{a(\xi_1^2+\xi_2^2)}{2\nu}\right)}{(1-R)^2}$$

$$R = \frac{\frac{\partial U_1}{\partial \xi_2}(0)}{2\nu}\int_0^{\xi_1}\exp\left(-\frac{a\xi^2}{2\nu}\right)d\xi\int_0^{\xi_2}\exp\left(-\frac{a\eta^2}{2\nu}\right)d\eta$$

or

$$\frac{\partial U_1}{\partial \xi_2} = -2\nu\frac{\partial^2}{\partial \xi_1 \partial \xi_2}\log(1-R)$$

3D

$$\frac{\partial^2 U_1}{\partial \xi_2 \partial \xi_3} = \frac{\partial^2 U_1}{\partial \xi_2 \partial \xi_3}(0) \exp\left(-\frac{a(\xi_1^2 + \xi_2^2 + \xi_3^2)}{2\nu}\right) \frac{1 - R}{(1 - R)^3}$$

$$R = \frac{\frac{\partial^2 U_1}{\partial \xi_2 \partial \xi_3}(0)}{2\nu} \int_0^{\xi_1} \exp\left(-\frac{a\xi^2}{2\nu}\right) d\xi \int_0^{\xi_2} \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta \int_0^{\xi_3} \exp\left(-\frac{a\zeta^2}{2\nu}\right) d\zeta$$

or

$$\frac{\partial^2 U_1}{\partial \xi_2 \partial \xi_3} = -2\nu \frac{\partial^3}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \log(1 - R)$$

Common: $\boldsymbol{U} = \nabla \phi,\ \phi = -2\nu \log(1 - R)$

2D Navier-Stokes equation

"A Mathematical Model Illustrating the Theory of Turbulence," (1948)

*Burgers vortex $\boldsymbol{u} = (u_r, u_\theta, u_\phi) = (-ar, v(r), 2az)$

$$\omega(r) = \frac{a\Gamma}{2\pi\nu} \exp\left(-\frac{ar^2}{2\nu}\right), \quad \Gamma \equiv \int_{\mathbb{R}^2} \omega_0(\boldsymbol{x}) d\boldsymbol{x}$$

$$v(r) = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{ar^2}{2\nu}\right)\right)$$

*Oseen vortex $\omega(r,t) = \dfrac{\Gamma}{4\pi\nu t} \exp\left(-\dfrac{r^2}{4\nu t}\right), \quad \omega(r,0) = \Gamma\delta(\cdot)$

If $\omega_0 \in L^2$, for $1 \leq p \leq \infty$,

$$t^{1-\frac{1}{p}} \left\| \omega(\boldsymbol{x}, t) - \frac{1}{2at} \Omega(\boldsymbol{\xi}) \right\|_p \to 0 \text{ as } t \to \infty$$

$$\Omega(\xi) = \frac{a\Gamma}{2\pi\nu} \exp\left(-\frac{a|\boldsymbol{\xi}|^2}{2\nu}\right), \boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2at}}$$

$$\frac{1}{2at}\Omega(\xi) = \frac{\Gamma}{4\pi\nu t} \exp\left(-\frac{|\boldsymbol{x}|^2}{4\nu t}\right)$$

$$\lim_{t\to 0} \Omega(\cdot) = \Gamma\delta(\boldsymbol{x}), \quad \Gamma \equiv \int_{\mathbb{R}^2} \omega_0(\boldsymbol{x}) d\boldsymbol{x}$$

e.g. Gallay & Wayne(2005)

The below are all 3D Navier-Stokes equations. $(\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{y})$

$$\frac{\partial \psi}{\partial t} = \frac{3}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{r} \times (\nabla \times \boldsymbol{\psi}(\boldsymbol{y}))\, \boldsymbol{r} \cdot (\nabla \times \boldsymbol{\psi}(\boldsymbol{y}))}{|\boldsymbol{r}|^5} \, \mathrm{d}\boldsymbol{y} + \nu \triangle \psi + a(\boldsymbol{x} \cdot \nabla) \psi$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u} + a(\boldsymbol{x} \cdot \nabla) \boldsymbol{u} + a\boldsymbol{u}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \triangle \boldsymbol{\omega} + a(\boldsymbol{x} \cdot \nabla) \boldsymbol{\omega} + 2a\boldsymbol{\omega}$$

$$\frac{\partial \boldsymbol{\chi}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\chi} + 2(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}) + \nu \triangle \boldsymbol{\chi} + \underbrace{a(\boldsymbol{x} \cdot \nabla)\boldsymbol{\chi} + 3a\boldsymbol{\chi}}_{= a\nabla \cdot (\boldsymbol{x} \otimes \boldsymbol{\chi})}$$

3. Navier-Stokes equations and Hopf equations (interlude)

3D Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \triangle \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0,$$

Property 1: if $\boldsymbol{u}(\boldsymbol{x}, t)$ is a solution, so is $\lambda \boldsymbol{u}(\lambda \boldsymbol{x}, \lambda^2 t)$.

Dynamic scaling

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{\sqrt{2at}} \boldsymbol{U}(\boldsymbol{\xi}, \tau),$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2at}}, \; \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a} \log(1 + 2at),$$

the scaled Navier-Stokes equations (the Leray equations)

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} = -\nabla_{\boldsymbol{\xi}} P + \nu \triangle_{\boldsymbol{\xi}} \boldsymbol{U} + a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}), \quad \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{U} = 0.$$

Characteristic functional of velocity $\Phi[\boldsymbol{\theta}, t] = \left\langle \exp\left(i \int \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{\theta}(\boldsymbol{x}) d\boldsymbol{x} \right) \right\rangle$

Hopf equation (1952)

$$\frac{\partial \Phi}{\partial t} = i \int d\boldsymbol{x}\, \theta_j^{\perp}(\boldsymbol{x}) \frac{\partial}{\partial x_k} \frac{\delta^2 \Phi}{\delta\theta_j(\boldsymbol{x})\delta\theta_k(\boldsymbol{x})} + \nu \int d\boldsymbol{x}\, \theta_j(\boldsymbol{x}) \triangle \frac{\delta \Phi}{\delta\theta_j(\boldsymbol{x})}, \quad \equiv L\Phi$$

$$\left(\frac{\partial}{\partial t} - L \right) \Phi \left[\lambda^2 \boldsymbol{\theta}(\lambda \boldsymbol{x}), \lambda^{-2} t \right] = 0, \quad \text{Rosen(1982)}$$

Extends to d-dimensions, hence,

Property 2: if $\Phi[\boldsymbol{\theta}(\boldsymbol{x}), t]$ is a solution, so is $\Phi\left[\lambda^{d-1}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t\right]$.

For vorticity char. functional, $\Phi\left[\lambda^{d-2}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t\right]$

For vorticity gradient char. functional, $\Phi\left[\lambda^{d-3}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t\right]$

See also Foias(1972,1973) for an alternative formulation based on the Liouville equation.

4. Self-similar solutions of Navier-Stokes equations

3D Navier-Stokes equations written in $\boldsymbol{\chi}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{\omega}$

$$\frac{\partial \boldsymbol{\chi}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\chi} + 2(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}) + \nu \triangle \boldsymbol{\chi}$$

After scaling, $\boldsymbol{X}(\boldsymbol{\xi}, \tau)$ satisfies:

$$\frac{\partial \boldsymbol{X}}{\partial \tau} = \nabla \times (\boldsymbol{U} \times \boldsymbol{X} + 2(\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{U}) + \underbrace{a\nabla \cdot (\boldsymbol{\xi} \otimes \boldsymbol{X})}_{=3a\boldsymbol{X} + a(\boldsymbol{\xi} \cdot \nabla)\boldsymbol{X}} + \nu \triangle \boldsymbol{X}$$

$$\boldsymbol{\chi} = \frac{\boldsymbol{X}(\boldsymbol{\xi})}{(2at)^{3/2}},\ \boldsymbol{\omega} = \frac{\boldsymbol{\Omega}(\boldsymbol{\xi})}{2at},\ \boldsymbol{u} = \frac{\boldsymbol{U}(\boldsymbol{\xi})}{(2at)^{1/2}}$$

$$\begin{cases} \boldsymbol{X}(\boldsymbol{\xi}) = \epsilon \boldsymbol{X}_1(\boldsymbol{\xi}) + \epsilon^2 \boldsymbol{X}_2(\boldsymbol{\xi}) + \dots \\ \boldsymbol{U}(\boldsymbol{\xi}) = \epsilon \boldsymbol{U}_1(\boldsymbol{\xi}) + \epsilon^2 \boldsymbol{U}_2(\boldsymbol{\xi}) + \dots \\ \boldsymbol{\Omega}(\boldsymbol{\xi}) = \epsilon \boldsymbol{\Omega}_1(\boldsymbol{\xi}) + \epsilon^2 \boldsymbol{\Omega}_2(\boldsymbol{\xi}) + \dots \end{cases}$$

Homogeneous Fokker-Planck equation

$$O(\epsilon): \quad \triangle \boldsymbol{X}_1 + \frac{a}{\nu} \nabla \cdot (\boldsymbol{\xi} \otimes \boldsymbol{X}_1) = 0$$

$$\boldsymbol{X}_1 = \int \boldsymbol{X}_0(\boldsymbol{\eta}) G(\boldsymbol{\xi} - \boldsymbol{\eta}) d\boldsymbol{\eta} = \boldsymbol{X}_0 * G$$

where $G(\boldsymbol{\xi}) = \left(\frac{a}{2\pi\nu}\right)^{3/2} \exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right)$

Inhomogeneous Fokker-Planck equation

$$O(\epsilon^2): \quad \triangle \boldsymbol{X}_2 + \frac{a}{\nu}\nabla\cdot(\boldsymbol{\xi}\otimes\boldsymbol{X}_2) = -\frac{1}{\nu}\nabla\times\underbrace{\{\boldsymbol{U}_1\times\boldsymbol{X}_1 + 2(\boldsymbol{\Omega}_1\cdot\nabla)\}\boldsymbol{U}_1}_{\equiv\boldsymbol{F}}$$

$\boldsymbol{U}_1 = \boldsymbol{U}_0 * G,\ \boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_0 * G$

$$\boldsymbol{X}_2 = -\frac{1}{\nu} g * (\nabla\times\boldsymbol{F})$$

$$= \frac{-1}{4\pi\nu}\left(\frac{a}{\nu}\exp\left(-\frac{ar^2}{2\nu}\right)\int_0^r \exp\left(\frac{as^2}{2\nu}\right)ds - \frac{1}{r}\right)*\nabla\times\{\boldsymbol{U}_1\times\boldsymbol{X}_1 + 2(\boldsymbol{\Omega}_1\cdot\nabla)\boldsymbol{U}_1\}$$

where g is the Green's function for the Fokker-Planck operator

$$g \equiv \frac{1}{4\pi}\left(\frac{a}{\nu}e^{-\frac{ar^2}{2\nu}}\int_0^r e^{\frac{as^2}{2\nu}}ds - \frac{1}{r}\right)$$

$$\boldsymbol{X}_2 = -\frac{1}{\nu}\underbrace{g*}_{L^2}\underbrace{\nabla\times}_{L^{-1}}\left\{\underbrace{\boldsymbol{U}_1}_{L^2}\times\boldsymbol{X}_1 + 2(\underbrace{\boldsymbol{\Omega}_1}_{L}\cdot\underbrace{\nabla)\boldsymbol{U}_1}_{L}\right\}$$

cf.

3D Burgers

$$\frac{\partial^2 U_1}{\partial\xi_2\partial\xi_3} = \frac{\partial^2 U_1}{\partial\xi_2\partial\xi_3}(0)\exp\left(-\frac{a(\xi_1^2+\xi_2^2+\xi_3^2)}{2\nu}\right)\frac{1-R}{(1-R)^3}$$

$$R = \frac{\frac{\partial^2 U_1}{\partial\xi_2\partial\xi_3}(0)}{2\nu}\underbrace{\int_0^{\xi_1}\exp\left(-\frac{a\xi^2}{2\nu}\right)d\xi}_{L}\underbrace{\int_0^{\xi_2}\exp\left(-\frac{a\eta^2}{2\nu}\right)d\eta}_{L}\underbrace{\int_0^{\xi_3}\exp\left(-\frac{a\zeta^2}{2\nu}\right)d\zeta}_{L}$$

3D Navier-Stokes

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} \left\| \boldsymbol{\chi}(\boldsymbol{x},t) - \frac{\boldsymbol{X}_1(\boldsymbol{\xi})}{(2at)^{3/2}} \right\|_p \to 0 \text{ as } t \to \infty$$

$$\text{Note}: \quad \frac{\boldsymbol{X}_1(\boldsymbol{\xi})}{(2at)^{3/2}} \to \boldsymbol{X}_0 * \delta = \boldsymbol{X}_0(\cdot) \text{ as } t \to 0,$$

where $\boldsymbol{X}_0(\boldsymbol{\xi})$ is singular, like the Dirac mass.

Initial data

$$\boldsymbol{X}_0 = \boldsymbol{P}\boldsymbol{c}\delta = (\boldsymbol{I} - \triangle^{-1}\nabla\nabla\cdot)\boldsymbol{c}\delta$$

$$= (\delta_{ij} - \triangle^{-1}\partial_i\partial_j)c_j\delta$$

$\boldsymbol{P}$=Leray projection, $\boldsymbol{c}$=const.

$$\boldsymbol{X}_1 = \boldsymbol{X}_0 * G$$

$$= (\delta_{ij} - \triangle^{-1}\partial_i\partial_j)c_j\delta * G$$

$$= (\delta_{ij} - \triangle^{-1}\partial_i\partial_j)c_j G$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\triangle^{-1} f = \frac{\delta_{ij}}{3} f + T_{ij}[f]$$

$$T_{ij}[f](\boldsymbol{x}) \equiv \frac{1}{4\pi}\mathsf{PV}\int \left(\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{y}|^3} - \frac{3(x_i - y_i)(x_j - y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^5} \right) f(\boldsymbol{y})\mathsf{d}\boldsymbol{y}$$

$$(\boldsymbol{X}_1)_i = \frac{2}{3} c_i G - T_{ij}[c_j G],$$

$$(\boldsymbol{X}_1)_i = c_i G + R_i R_j c_j G$$

or,

$$\boldsymbol{X}_1 = \frac{2}{3}\boldsymbol{c}G - \frac{1}{4\pi}\mathsf{PV}\int \left(\frac{\boldsymbol{c}G(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3} - \frac{3(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{c}G(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^5} \right) \mathsf{d}\boldsymbol{y}$$

Step 1:

$$\boldsymbol{X} = \epsilon \boldsymbol{X}_1 + \epsilon^2 \boldsymbol{X}_2 + \ldots$$

or, component-wise

$$X = \epsilon X_1 \left(1 + \epsilon \frac{X_2}{X_1} + \ldots \right)$$

In practice, rationalisation in the spirit of Padé approximation may be useful

$$X \approx \frac{X_1}{1 - \frac{X_2}{X_1}}$$

(for step 2) corresponding general heat flow

$$\boldsymbol{\chi}_1 = \int \boldsymbol{\chi}_0(\boldsymbol{y})\widehat{G}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y} = \boldsymbol{\chi}_0 * \widehat{G}, \;\; \boldsymbol{u}_1 = \boldsymbol{u}_0 * \widehat{G},\; \boldsymbol{\omega}_1 = \boldsymbol{\omega}_0 * \widehat{G}$$

$$\widehat{G} = \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-\frac{|\boldsymbol{x}|^2}{4\nu t}\right)$$

$$\boldsymbol{\chi}_2 = \frac{-1}{4\pi\nu}\left(\frac{a}{\nu}\exp\left(-\frac{ar^2}{2\nu}\right)\int_0^r \exp\left(\frac{as^2}{2\nu}\right)ds - \frac{1}{r}\right) * \nabla\times\{\boldsymbol{u}_1\times\boldsymbol{\chi}_1 + 2(\boldsymbol{\omega}_1\cdot\nabla)\boldsymbol{u}_1\}$$

$$\boldsymbol{\chi} = \epsilon\boldsymbol{\chi}_1 + \epsilon^2\boldsymbol{\chi}_2 + \ldots$$

Component-wise

$$\chi = \epsilon\chi_1\left(1 + \epsilon\frac{\chi_2}{\chi_1} + \ldots\right), \;\; \chi \approx \frac{\chi_1}{1 - \frac{\chi_2}{\chi_1}}$$

1D In $U_1 = U_0 * G$, $U_0 = M\delta \;\Rightarrow\; U_1 = MG$

$$\frac{U(\xi)}{\sqrt{2at}} \to M\delta, \quad M = \int u_0.$$

2D In $\Omega_1 = \Omega_0 * G$, $\Omega_0 = \Gamma\delta \;\Rightarrow\; \Omega_1 = \Gamma G$

$$\frac{\Omega(\xi)}{2at} \to \Gamma\delta, \quad \Gamma = \int \omega_0.$$

3D In $X_1 = X_0 * G$, $X_0 = K\boldsymbol{P}\delta\boldsymbol{c} \Rightarrow X_1 = K\boldsymbol{P}G\boldsymbol{c}$

$$\frac{X(\xi)}{(2at)^{3/2}} \to K\boldsymbol{P}\delta\boldsymbol{c}, \quad K\boldsymbol{c} = \int \nabla \times \boldsymbol{\omega}_0 = \int d\boldsymbol{S} \times \boldsymbol{\omega}_0$$

5. Summary and outlook

Self-similar solutions of the Navier-Stokes equations

Two kinds of scale-invariance

d-dimensional Burgers equations

3D Navier-Stokes equations:
asymptotic form of self-similar decaying solutions

Numerical evaluations ? Implications on statistical solutions ?