Random dynamical systems for stochastic pde driven by an fractional Brownian motion with application to the stochastic shell model

Part I RDS, FBM, SPDE driven by FBM

Part II the stochastic shell model.

1 Noise and RDS

Noise $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ canonical probability space

- Ω set of noise trajectories
- θ measurable flow on Ω

$\theta : \mathbb{R} \times \Omega \to \Omega$

$\theta_t \circ \theta_\tau = \theta_{t+\tau}$, $\theta_0 = id_\Omega$

$\theta_t \mathbb{P} = \mathbb{P}$ (+ ergodicity) noise is stationary

Examples

- Brownian motion $\mathcal{P}_{BM}$

$$\Omega = C_0(\mathbb{R}, \mathcal{U}) \; , \; \mathbb{P} = \mathbb{P}_{Wiener}$$ (distribution of a Gauß process)

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$$
$\omega \in \Omega$

- stationary Ornstein Uhlenbeck process $\mathcal{P}_{OU}$

$$\Omega = C(\mathbb{R}, \mathcal{U}) \quad \mathbb{P}_{OU}$$

$$\theta_t z(\cdot) = z(\cdot + t) \quad z \in \Omega$$

- Lévy noise $\mathcal{P}_L$ (jump noise)

$$\Omega = \mathcal{D}_0(\mathbb{R}, \mathcal{U}) \; , \; \mathbb{P} = \mathbb{P}_L$$

- <u>Fractional Brownian motion</u> $\mathcal{P}_{FBM_H}$ <u>$H \in (0,1)$</u>

$$\Omega = C_0(\mathbb{R}, \mathcal{U})$$

$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$, $\mathbb{P}_{FBM_H}$ Gauß measure

<u>covariance</u> $R(s,t) = \frac{1}{2} Q \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right)$

Lemma (good / bad properties of FBM)

- $H \neq \frac{1}{2}$ $\mathcal{P}_{FBM_H}$ is not a martingale,
  no independent increments
  no Markov property
  $\Rightarrow$ Ito theory fails

- $H = 1/2$ $\mathcal{P}_{FBM_{1/2}} = \mathcal{P}_{BM}$

- $H \in (\frac{1}{2}, 1)$ $\mathcal{P}_{FBM_H}$ has a "long term memory"

$$\Sigma_{FBM_H} = \sum_{i=1}^{\infty} \operatorname{cov}(\underbrace{\omega(1)-\omega(0)}_{\theta_0 \omega(1)}, \underbrace{\omega(i+1)-\omega(i)}_{\theta_i \omega(1)}) = \infty$$

$(\Sigma_{BM} = 0)$

- $\omega \in C^{H^-}$ ~ β Hölder continuous $\forall \beta < H$.

- $\mathbb{P}_{FBM_H} = \theta_t \mathbb{P}_{FBM_H}$

$$\mathbb{E}(\theta_t \omega(s)\, \theta_t \omega(t)) = R(s,t)$$

$\mathbb{P}_{FBM_H}$ is θ-ergodic

2. Random dynamical Systems

Noise $\mathbb{P}$

$\varphi : \mathbb{R}^+ \times \Omega \times V \to V$ measurable

V sep. Hilbertspace

$\varphi(t+\tau, \omega, u_0) = \varphi(t, \theta_\tau \omega, \cdot) \circ \varphi(\tau, \omega, u_0)$

$\forall\ t, \tau \geq 0,\ u_0 \in V$

$\varphi(0, \omega, \cdot) = id_V$ cocycle property.

$\left.\right\} \forall \omega \in \Omega_0 \in \mathcal{F}$, $\theta_t \Omega_0 = \Omega$, $\mathbb{P}(\Omega_0) = 1$

Remark: • Almost surely is not allowed.

• RDS $\Rightarrow$ Existence of random attractors

random invariant manifolds

exponential stability ...

Examples

• $\frac{du}{dt} = Au + F(\theta_t \omega, u)$ $\quad u(0) = u_0 \in V$ generates an RDS

• $du = F(u)dt + G(u)dw_{BM}$ $\quad V = \mathbb{R}^d$, $u(0) = u_0 \in \mathbb{R}^d$

• $du = Au\,dt + F(u)dt + G(u)\,dw_{BM}$ $\quad \dim V = \infty$ $\quad$ RDS ??

(Ito theory)

Ito integral $\quad \int_0^t G(u_0(\tau, \omega))dw \in \underbrace{L_2(\Omega)}_{\text{equivalence classes}}$

- Why finite dimensional Ito equations generate an RDS?

Solution

$u_{u_0}(t) = X(t, u_0)$ exists almost surely

$\forall t \geq 0, \forall x \in \mathbb{R}^d$

There exists a random field $\tilde{X}(t, u_0)$

$\tilde{X}(t, u_0) = X(t, u_0)$ almost surely $\forall t \geq 0 \; u_0 \in \mathbb{R}^d$

$(t, u_0) \mapsto \tilde{X}(t, u_0)$ is (Hölder cont.)

$\tilde{X} \longrightarrow \varphi$ (RDS)

- RDS for spde

$$du = (\nu A u + B(u,u) + f)dt + \begin{cases} dw \\ u \circ dw \end{cases} \; \mathbb{R}^1$$

$$dz = \begin{cases} \nu A z \, dt + dw \\ -z \, dt + dw \end{cases}$$

$$\frac{dv}{dt} = (\nu A v + B(v,v) + B(v, Z(\theta_t \omega)) + B(Z(\theta_t \omega), v) + B(Z(\theta_t \omega), Z(\theta_t \omega)) + f$$

generates an RDS, random attractors, ...

Flandoli 199?, Imkeller, S 200?

3. The Pathwise Integral

Def (fractional derivative)

$$D^{\alpha}_{a+} f[r] = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(r)}{(r-a)^{\alpha}} + \alpha \int_a^r \frac{f(r)-f(q)}{(r-q)^{\alpha}} dq \right)$$

$$D^{1-\alpha}_{b-} \omega_{b-}[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\omega(r)-\omega(b)}{(b-r)^{1-\alpha}} + (1-\alpha) \int_r^b \frac{\omega(r)-\omega(q)}{(q-r)^{2-\alpha}} dq \right)$$

sufficient conditions for existence f, ω Hölder

Def (The pathwise integral, Young integral)

$$\int_a^b f\, d\omega = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f[r]\, D^{1-\alpha}_{b-} \omega_{b-}[r]\, dr$$

(generalized integration by parts formula)

Theorem (Existence)

- Assume $\omega \in C^{\beta'}$, $f \in C^{\beta}$

  $\alpha < \beta$, $\beta + \beta' > 1$ Then the pathwise Integral exists

- $(\omega, f) \mapsto \int_a^b f\, d\omega$ bilinear + continuous

- $a < b < c$ $\int_a^c \cdots = \int_a^b \cdots + \int_b^c$

- $\int_{a+t}^{b+t} f(r)\, d\omega = \int_a^b f(r+t)\, d\theta_t \omega[r]$ $\boxed{\text{RDS}}$

Remark (history)

- Weyl, Liouville, Young, ...
- Samko et al.
- Zähle (stochastic integrals)
- Nualart Rașcanu, Maslowski Nualart sode, spde

$\mathcal{P}_{FBM_H}$ $H > \frac{1}{2}$

$$du = F(u)\,dt + G(u)\,d\omega_{FBM_H} \qquad V = \mathbb{R}^d$$

$$du = Au\,dt + F(u)\,dt + G(u)\,d\omega_{FBM} \qquad V = HS$$

Theorem (Maslowski, Nualart; Gao, Lu, S)

- A generates a C_0 analytic semigroup S
- $F \in C_b^1$
- $G \in C_b^1, C_b^2$

Then there exists a unique mild solution $\forall \omega \in \Omega_0$:

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))\,dr + \underbrace{\int_0^t S(t-r)G(u(r))\,d\omega(r)}_{\leq c \|S(t-\cdot)G(u(\cdot))\|_\beta \, |||\omega|||_{\beta'}}$$

- The solution generates an RDS.
- The pathwise integral is Hölder cont.

The case $\mathbb{P}_{FBM_H}$ $H \in (1/3, \frac{1}{2}]$

- includes $\mathbb{P}_{BM}$
- Young integral does not work
- Rough path theory

Lyons, Gubinelli

SEE: Gubinelli Tindel, Deya Gubinelli Tindel

"Hesse, Neamțu"

- alternative theory: Hu Nualart, Garrido, Lu, S.

compensated fractional derivatives

$$u(t) = S(t)u_0 + (-1)^{\alpha} \int_0^t \hat{D}^{\alpha}_{0+} S(t-r) G(u)[r] D^{1-\alpha}_{t} \omega_t[r]\, dr$$

$$+ (-1)^{2\alpha-1} \int_0^t \hat{D}^{2\alpha-1}_{0+} DG(u)[r] D^{1-\alpha}_t \mathcal{D}^{1-\alpha}_t v[r]\, dr$$

$$\hat{D}^{\alpha}_{0+} G(u)[r] = \frac{1}{\Gamma(1-\alpha)} \Big(\frac{G(u(r))}{r^{\alpha}} + \alpha \int_0^r \frac{\overbrace{G(u(r)) - G(u(q)) - DG(u(q))(u(r)-u(q))}^{\sim D^2 G(u(q)) \overbrace{(u(r)-u(q))^2}^{|||u|||_{\beta}^2 (r-q)^{2\beta}}}}{(r-q)^{1+\alpha}}\, dq$$

Theorem

- $G \in C_b^i \quad i = 1,2,3$
- $F \in C_b^1$, ...

Then there exists a solution, unique.

generating an RDS

$V = u \otimes \omega = \int_s^t (u(r) - u(s)) \otimes d\omega$ i.g. does not exist.

formulate a second equation for V.

- $U = (u, v)$

$$\underline{U = \mathcal{T}(U, u_0, \omega)} \text{ in } W_{\beta,[0,T]}$$

$$\|\mathcal{T}(U, u_0, \omega)\|_{W_\beta} \le C(\|u_0\| + T^\beta(1 + |||\omega|||_\beta)(1 + \underline{\underline{|||U|||^2_{W_\beta}}}))$$

Dynamics:

local exponential stability

- A generates an exponential stable semigroup, only local
- $G(0) = 0,\ DG(0) = 0,\ D^2G(0) = 0$

- Summary

- Theory for EE with non absolutely cont. integrator $d\omega \neq \omega' dt$

- Application for $\mathbb{P}_{FBM_H}$

- $H \neq 1/2$ there is no Markov dynamics

- Application of RDS theory.

- deterministic theory needs random input

ω β' Hölder Gauß process

$$\omega \otimes \omega(s,t) = \int_s^t (\omega(r) - \omega(s)) \otimes d\omega(r) \sim 2\beta' \text{ Hölder}$$

Infinite dimensional situation

$$\omega \otimes_S \omega(s,t) = \int_s^t \int_s^{\xi} S(\xi - r) \circ d\omega(r) \otimes d\omega(\xi).$$

4 The Stochastic Shell Model (Bessai, Garrido, S)

mathematical setting: Constantin, Levant, Titi

Bessaih, Ferrario

- A positive symmetric (unbounded) operator on Hilbert space $V = V_0$ with compact inverse
  - generates a complete ONS (k_n^2, e_n) $(k_1^2 \leq k_2^2 \dots)$
  - generates an analytic semigroup.
- $B: V_{1/2} \times V \to V$, $V \times V_{1/2} \to V$ cont. $(B(u,v), v) = 0$

$d_1 + d_2 + d_3 \geq 1$ $\quad d_i \in \mathbb{R}$

$$\|B(u,v)\|_{V_{-d_3}} \leq C \|u\|_{V_{d_1}} \|v\|_{V_{d_2}} \quad u \in V_{d_1}, v \in V_{d_2}$$

- $\mathbb{P}_{FBM_H}$ $\quad H \in (1/2, 1)$
- $G: V_\delta \to L_2(V, V)$ $\quad \delta > 0 \dots$

$G \in C_b, C_b^1, C_b^2$

...

We are looking for

Def (mild solution)

$$u \in C([0,T], V) \cap L_2(0,T,V) \cap C^{\beta}([0,T], V_{-\delta}), \ \delta > 0$$

$$u(t) = S(t)u_0 + \int_0^t S(t-r) B(u(r), u(r))\, dr$$
$$+ \underbrace{\int_0^t S(t-r) G(u)(r)\, d\omega}_{\text{pathwise integral}}$$

Existence (strategy)

$\omega_n \xrightarrow{C^{\beta'}} \omega$ piecewise linear approximation

$$\omega_n(t) = \int_0^t \underbrace{\frac{\omega(r+\frac{1}{n}) - \omega(r)}{1/n}}_{\theta_r\omega(1/n)/\,1/n} dr \qquad \text{RDS approx.}$$

Approximate problems for 'smooth' ω.

$$\frac{du}{dt} = Au + B(u,u) + G(u)\,\dot{\omega}_n(t) \qquad *$$

Lemma:

Assume A, B, G

Then $*$ has a unique weak solution satisfying the energy (in)equality

$$\boxed{\|u(t)\|^2 + \int_0^t \|u(r)\|^2_{V_{1/2}}\, dr \le \|u_0\|^2 + C |||\omega|||^{\beta+p'}_{\beta'} (1 + |||u|||_C) |||u|||_{\beta,-\delta}.}$$

Ideas of the proof

- $\| D_{t-}^{1-\alpha} \omega_{t}[\tau] \| \leq C |||\omega|||_{\beta'} (t-\tau)^{\alpha+\beta'-1}$
- $\left| \int_0^t (G^*(u(\tau)) u(\tau), \omega'(\tau))_V \, d\tau \right|$

$$\leq C |||\omega|||_{\beta'} \int_0^t \underbrace{(t-\tau)^{\alpha+\beta'-1}} \left(\frac{\|G^*(u(\tau)) u(\tau)\|}{\tau^\alpha} + \int_0^\tau \frac{\|G^*(u(\tau)) u(\tau) - G^*(u(q)) u(q)\|}{(\tau-q)^{1+\alpha}} dq \right) d\tau$$

$\|G^*(u(\tau)) u(\tau)\| \leq C \|u\|_C \cdots$

$$\int_0^\tau \frac{\|G^*(u(\tau)) u(\tau) - G^*(u(q)) u(q)\|}{(\tau-q)^{1+\alpha}} dq$$

$$\leq C_G \int_0^\tau \frac{\|u(\tau) - u(q)\|_{V_{-\delta}}}{(\tau-q)^{1+\alpha}} dq + \|u\|_C \underbrace{\int_0^\tau \frac{\overbrace{\|G^*(u(\tau)) - G^*(u(q))\|}^{\overbrace{C_{DG} \|u(t) - u(q)\|_{V_{-\delta}}}^{\leq C_{DG} |||u|||_{\beta,\delta} (\tau-q)^\beta}}}{(\tau-q)^{1+\alpha}} dq}_{\tau^{\beta-\alpha}}$$

$$\leq c(C_G + C_{DG} \|u\|_C) |||u|||_{\beta,-\delta}$$

$$\int_0^t (G^*(u(\tau)) u(\tau), \omega'(\tau)) d\tau \leq C |||\omega||| t^{\beta'} \|u\|_C + C |||\omega|||_{\beta'} t^{\beta+\beta'} (1 + \|u\|_C) |||u|||_{\beta,\delta}$$

We see: we also need an estimate for $|||u|||_{\beta,-\vartheta}$

$$A^{-\vartheta}(u(q)-u(p)) = A^{-\vartheta}(S(q)-S(p))u_0$$
$$+ A^{-\vartheta+1/2}\int_p^q S(q-r)\,A^{-1/2}B(u(r),u(r))\,dr$$
$$+ A^{-\vartheta+1/2}\int_0^p (S(q-r)-S(p-r))A^{-1/2}B(u(r),u(r))\,dr$$
$$+ \dots + \dots \quad = I_1 + \dots + I_5$$

Trouble maker S:

$$\|I_1(p,q)\| \le \frac{\|A^{-\vartheta}(S(q-p)-id)\,S(p)u_0\|}{(q-p)^\beta}$$
$$\le \frac{(q-p)^\vartheta\,\|A^{-\vartheta}A^{\vartheta}S(p)\,u_0\|}{(q-p)^\beta} \le C\|u_0\|\,t^{\vartheta-\beta}$$

$$\frac{\|I_3(p,q)\|}{(q-p)^\beta} \le \frac{1}{(q-p)^\beta}\int_0^p \|A^{-\vartheta+1/2}\overbrace{(S(q-p)-id)}^{(q-p)^\beta}\,\overbrace{S(p-r)}^{(p-r)^{-1/2+\vartheta-\beta}}\,\times$$
$$\times A^{-1/2}B(u(r),u(r))\|\,dr$$
$$\le C\|u\|_C^2\underbrace{\int_0^p (p-r)^{\vartheta-\frac{1}{2}-\beta}\,dr}_{\le C\,t^{1-\beta}}$$

$$\boxed{|||u|||_{\beta,-\vartheta} \le C\,t^{\vartheta-\beta}\,\|u_0\| + C\|u\|_C^2\,t^{1-\beta} + |||u|||_{\beta'}\,t^{\beta'-\beta}\,(1+t^\beta\,|||u|||_{\beta,-\vartheta})}$$

Can be solved for small t

Remark (Aubin, Dubinski compactness theorem)

- $L_2(0,T; V_2) \cap C^\beta([0,T], V_{-\delta}) \underset{c}{\subseteq} L_2(0,T, V) \cap C([0,T], V_{-\delta})$
- $0 < \delta_1 < \delta_2,\ \beta_1 < \beta_2 \leq 1 \quad C^{\beta_2}([0,T], V_{-\delta_1}) \underset{c}{\subseteq} C^{\beta_1}([0,T], V_{-\delta_2})$

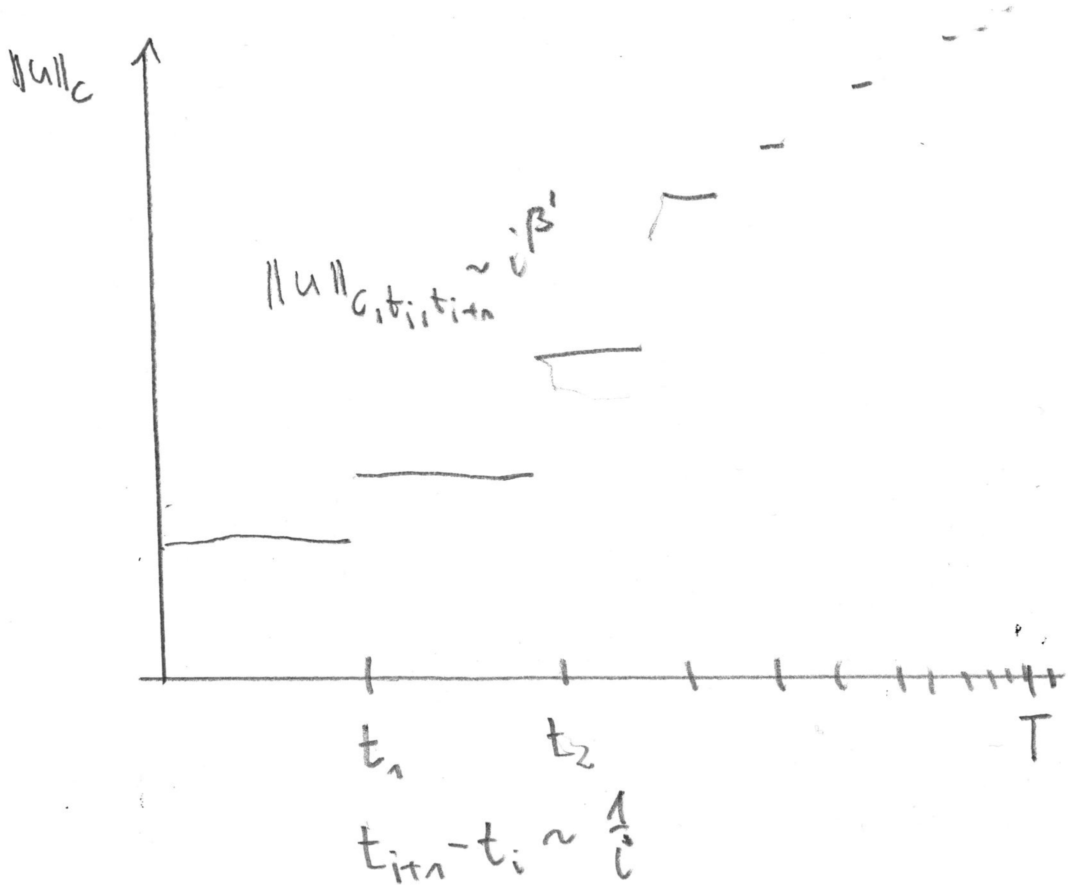
<u>Theorem (Main)</u>

- 1. the stochastic shell model has a local solution
- 2. This solution is unique
- The solution can be extended to any interval $[0,T]$
- The solution generates an RDS

"Proof" $\quad \{u(\cdot, \omega_n)\ n \in \mathbb{N}\} \quad \omega_n \xrightarrow{C_r} \omega$

- $L_2(0,T, V_{1/2})$ weak compact
- $L_2(0,T,V) \cap C([0,T], V_{-\delta})$ compact
- $C^\beta([0,T], V_{-\delta})$ compact

The limit points are mild solutions.



no explosion

<u>Question</u>: Random attractors?

Thank You!