Random perturbations of predominantly hyperbolic systems

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New Developments in Open Dynamical Systems and Their Applications

Joint work with J. Xue, Y. Yang, L.-S. Young

Consider a smooth map $f : M \to M$.

Definition (Informal)

We say f is **predominantly expanding (resp. hyperbolic)** if there exists $\mathcal{C} \subset M$ (possibly noninvariant) such that

- $f|_{M \setminus \mathcal{C}}$ is uniformly expanding (resp. hyperbolic), and
- $\mathcal{C}$ is "small" (e.g., $\text{Leb}(\mathcal{C}) \ll 1$).

Question

What is the **asymptotic dynamical regime** of f?

Examples I have in mind: dynamics on $\mathcal{C}$ *harms/reverses* hyperbolicity.

(a) 1D maps with critical points, e.g. quadratic family $q_c(x) = x^2 + c, c \in [-\frac{1}{4}, 2]$, $\mathcal{C}$ = neighborhood of 0
   - Derivative growth reversed near critical point

(b) 2D maps with 'cone twisting', e.g., Standard map family w. large parameter: $\mathcal{C}$ = two thin strips
   - Critical strip 'twists' unstable cone towards contracting directions.

(a) 1D maps with critical points, e.g. quadratic family $q_c(x) = x^2 + c, c \in [-\frac{1}{4}, 2]$, $\mathcal{C}$ = neighborhood of 0

- Existence of a.c.i.m., positive exponent under (typically) **uncheckable** infinite-time conditions

(b) 2D maps with 'cone twisting', e.g., Standard map family w. large parameter: $\mathcal{C}$ = two thin strips

- Open whether standard map has positive exponent on positive volume set ("stochastic sea")

Subject of this talk: add small IID random perturbations to such systems at each timestep.

- Main idea: sufficient amount of randomness "shakes loose" hyperbolicity
- Part 1: 1D dynamics with arbitrarily small noise amplitudes (joint with Yun Yang, in prep.)
- Part 2: 2D dynamics with sufficiently large perturbations
  - Standard map (B.-Xue-Young '17, Ann. Math.)
  - Possibly dissipative maps with 'Henon flavor' (B.-Xue-Young '17, CMP)

Part 1: One-dimensional dynamics

Chaos in 1D

1D: Prototypical example: family $f_a : S^1 \to S^1$,

$$f_a(x) = 10\sin(2\pi x) + a(mod 1)\,.$$

For which a is f_a chaotic? (i.e., a.c.i.m., positive exponent, decay of correlations)

- f_a is predominantly expanding away from neighborhood of $\{f_a' = 0\} = \{\frac{1}{4}, \frac{3}{4}\}$.
- Primary obstruction: formation of sinks when **postcritical orbit** $f_a^n(\hat{x}), \hat{x} \in \{\frac{1}{4}, \frac{3}{4}\}$ comes too close to $\hat{x}$.

Mechanism for chaos for map w critical points

$$f_a(x) = 10\sin(2\pi x) + a(mod 1)$$

A mechanism for positive Lypaunov exponents: assume $f = f_a$ satisfies Misiurewicz condition

$$(H) \qquad \min_{n\geq 1} \operatorname{dist}(f^n\{f' = 0\}, \{f' = 0\}) \geq c > 0\,.$$

Given $x \in S^1 \setminus \{f_a' = 0\}$, show $|(f_a^n)'(x)| \gtrsim e^{\lambda n}$:

- $f = f_a$ is predominantly expanding away from neighborhood $\mathcal{C}$ of $\{f' = 0\} = \{\frac{1}{4}, \frac{3}{4}\}$.
- If $x_i := f^i(x)$ falls in $\mathcal{C}$, then $|f'(x_i)| \ll 1$.
  - Orbit of x_i **shadows** postcritical orbit $\{f^n(\hat{x})\}$ for p timesteps, where $p \approx -\log|f_a'(x_i)|$ (a.k.a. "bound period")
  - (H) $\Rightarrow$ $f^j(\hat{x}) \in \{|f'| \geq 5\}$ for all j, hence $f^j(x_i) \in \{|f'| \geq 4\}$ for $j \leq p$.

Structural instability/ “comingled regimes”

$$f_a(x) = 10\sin(2\pi x) + a(mod 1)\,,$$

$$(H) \qquad \min_{n\geq 1} \mathrm{dist}(f^n \hat{x}, \{f' = 0\}) \geq c > 0\,.$$

- Condition (H) rules out sinks, but...
  - (H) not a stable property w.r.t. parameter a
  - (H) typically not checkable.
- Issues are real: for real quadratic family $q_c(x) = x^2 + c$, $c \in [-\frac{1}{4}, 2]$,
  - A.e. q_c is ‘regular’ (sinks) or ‘stochastic’ (a.c.i.m., positive exponent) ; see Lyubich ’97 and others
  - $\{c \in [0,1) : q_c$ has a sink$\}$ is **open and dense**
  - “Stochastic” parameters have convoluted Cantor-like structure, positive Lebesgue measure
    - E.g., Jakobson ’81, Benedicks & Carleson ’85 (these use a weaker form of (H))

Random perturbations

Introduce small, IID random perturbations at each timestep:

- Roughly speaking, chaotic regimes of such random dynamics tend to be **robust / structurally stable**, unlike deterministic dynamics
- Conceptually, should be possible to find **checkable** conditions for chaos
- Not so unnatural: real world inherently noisy!

Random perturbations

Our model:

$$f(x) = f_{a,L}(x) := L\sin(2\pi x) + a \quad (mod 1)$$

where $L \gg 1$ fixed and $a \in [0,1)$ a parameter.

Introduce IID random perturbations of small amplitude at each timestep:

- Fix "noise amplitude" $\epsilon > 0$
- IID $\omega_1, \omega_2, \cdots$ uniformly distributed in $[-\epsilon, \epsilon]$, $\underline{\omega} := (\omega_i)_{i \geq 1}$.

Given $f = f_{a,L}$, at time i perturb to $f_{\omega_i} = f(\cdot + \omega_i)$.

Question

For given $a \in [0,1)$, what is the asymptotic dynamical regime of

$$f^n_{\underline{\omega}} = f_{\omega_n} \circ \cdots \circ f_{\omega_1} \, ?$$

Focus on Lyapunov exponents: $\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log |(f^n_{\underline{\omega}})'(x)|$, when lim exists.

$$f = f_a : S^1 \to S^1, \quad f_{\omega_i} = f(\cdot + \omega_i) \quad f^n_{\underline{\omega}} = f_{\omega_n} \circ \cdots \circ f_{\omega_1}$$
$$f(x) = f_{a,L}(x) := L \sin(2\pi x) + a \quad (mod 1), L \gg 1$$

- Low period sinks (fixed, periodic) for f_a persist as **random sinks** for ϵ sufficiently small
- High period sinks **destroyed** by noise if ϵ large enough.

Given fixed $\epsilon > 0$, asymptotic regime of $(f^n_{\underline{\omega}})$ should depend on only **finitely many** iterates of $f = f_a$.

Question:

Given $\epsilon > 0$, how many iterates of $f = f_a$ determine asymptotic behavior of $(f^n_{\underline{\omega}})$?

Checkable, finite-time condition:

$$f = f_a : S^1 \to S^1, \quad f_{\omega_i} = f(\cdot + \omega_i) \quad f^n_{\underline{\omega}} = f_{\omega_n} \circ \cdots \circ f_{\omega_1}$$
$$f(x) = f_{a,L}(x) := L \sin(2\pi x) + a \quad (mod 1), L \gg 1$$

Sinks of period $\leq k$ ruled out when parameter a satisfies **finite-time Misiurewicz condition**

$$(H)_{c,k} \qquad \text{dist}(f^i \hat{x}, \{f' = 0\}) \geq c \text{ for all } 1 \leq i \leq k, \hat{x} \in \{f' = 0\}$$

for fixed $c > 0, k \in \mathbb{Z}_{\geq 1}$.

- $(H)_{c,k}$ is checkable! Satisfied by open set of a of mass $\approx (1-c)^k$.
- No assumptions made about $k+1$-th iterate. **Sink of period $k+1$ possible.**

Results:

$$f(x) = f_{a,L}(x) := L\sin(2\pi x) + a \quad (mod 1)$$

$$(H)_{c,k}: \quad \operatorname{dist}(f^i\hat{x}, \{f' = 0\}) \geq c \text{ for all } 1 \leq i \leq k,\ \hat{x} \in \{f' = 0\}$$

$$\text{Perturbations } f_{\omega_i} := f(\cdot + \omega_i), \quad \text{Markov chain } X_i := f_{\omega_i}(X_{i-1})$$

Fix $c > 0, \beta \in (0,1)$. Let $L \geq L_0$ where $L_0 = L_0(c, \beta)$.

Theorem

For any k and any $\epsilon \geq L^{-(2k+1)(1-\beta)+\beta}$, the following holds:

(i) *Markov chain (X_i) admits unique stationary measure μ, everywhere-supported (hence $\lambda = \lim_{n\to\infty} \frac{1}{n} \log |(f^n_{\underline{\omega}})'(x)|$ exists and is constant for all $x \in S^1$ w.p.1)*

(ii) $\lambda \geq \gamma_0 \log L$, *where*

$$\gamma_0 := \min\{\frac{(2k+1)(1-\beta) - \alpha}{k+1}, \frac{1}{2} - \beta\} \qquad \textit{and } \epsilon = L^{-\alpha}$$

$$f(x) = f_{a,L}(x) := L\sin(2\pi x) + a \quad (mod 1)$$

$$(H)_{c,k}: \quad \text{dist}(f^i\hat{x}, \{f'=0\}) \geq c \text{ for all } 1 \leq i \leq k,\ \hat{x} \in \{f'=0\}$$

$$\text{Perturbations } f_{\omega_i} := f(\cdot + \omega_i), \quad \text{Markov chain } X_i := f_{\omega_i}(X_{i-1})$$

Boundary $\epsilon = L^{-(2k+1)}$ is essentially sharp:

Proposition

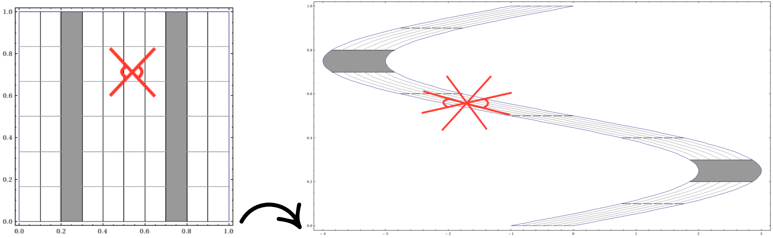
Assume f satisfies $(H)_{c,k}$, $f^{k+1}(\hat{x}) = \hat{x}$ for some $\hat{x} \in \{f'=0\}$ and $\epsilon = L^{-(2k+1)}$. Then, the Markov chain (X_i) has a stationary measure μ supported on a $\approx L^{-(k+1)}$-neighborhood of the orbit $\{f^i\hat{x}\}_{i=0}^k$. Moreover, the Lyapunov exponent of μ satisfies $\lambda \leq -\log 2$ (i.e., μ is a random sink).

Part II: 2D dynamics

2D dynamics

Parameters $b \in (0,1]$, $L \gg 1$. Consider (possibly discontinous) model $F = F_{b,L} : \mathbb{T}^2 \to \mathbb{T}^2$,

$$F(x,y) := (2x + L\sin(2\pi x) - y,\ bx)\,.$$



- For $L \gg 1$, F is predominantly hyperbolic with expansion $\sim L$ along x-axis on unshaded region
- Shaded region $\mathcal{C}$ is $O(L^{-1})$ neighborhood of $\{x = \frac{1}{4}, \frac{3}{4}\}$

Obstructions to hyperbolicity in 2D

$$F(x, y) := (2x + L\sin(2\pi x) - y,\ bx)$$

Estimating LE is a **delicate** cancellation problem:

- Growing vectors 'twisted' into contracting directions
- Conservative $(b = 1)$: elliptic islands
- Dissipative $(b < 1)$: presence of sinks of high period

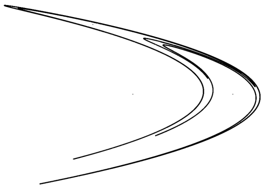
Obstructions are real:

- **Conservative:** For Chirikov standard map, proliferation of elliptic islands for large set of L (Duarte 95)
- **Dissipative:** coexistence of wild hyperbolic sets and infinitely many sinks (Newhouse 74)

Existing positive results

Conservative:

- (Gorodetski 12) Chirikov standard map: $\lambda_1 > 0$ on set of Hausdorff dimension 2 (zero volume)



Dissipative:

- Dynamics of Hénon map in (Benedicks & Carleson 91)
- One direction of instability (Wang & Young 01, 08)

Results entail **intensive** parameter exclusion to rule out bad behavior, e.g., formation of sinks.

Random perturbations

$$F(x, y) := (2x + L\sin(2\pi x) - y,\ bx)$$

Add IID random perturbations at each timestep:

- $\omega_i, i \geq 1$ IID, distributed uniformly in $[-\epsilon, \epsilon]$
- Perturb to $F_{\omega_i}(x, y) = F(x + \omega_i, y)$
- Compositions $F^n_{\underline{\omega}} = F_{\omega_n} \circ \cdots \circ F_{\omega_1}$; $\underline{\omega} = (\omega_1, \cdots, \omega_n)$.

For "large" ϵ, clear that top Lyapunov exponent $\lambda_1^\epsilon = \lim_{n\to\infty} \frac{1}{n} \log \|(dF^n_{\underline{\omega}})_x\|$ exists, $\lambda_1^\epsilon \sim \log L$.

Question

How large to take ϵ to "shake loose" hyperbolicity, i.e., $\lambda_\epsilon^1 \sim \log L$?

Results: volume-preserving ($b = 1$)

$$F(x,y) := (2x + L\sin(2\pi x) - y,\, x), \qquad F_{\omega_i}(x,y) = F(x+\omega_i, y),$$
$$F^n_{\underline{\omega}} = F_{\omega_n} \circ \cdots \circ F_{\omega_1}$$

Theorem (Joint with JX, LSY; Ann. Math. 2017)

There exists $L_0, c > 0$ such that for any $L \geq L_0$ and

$$\epsilon > L^{-cL^{9/10}},$$

the top Lyapunov exponent $\lambda_1^\epsilon(p) = \lim_{n\to\infty} \frac{1}{n} \log \|(dF^n_{\underline{\omega}})_p\|$ exists, is almost surely constant over $p, \underline{\omega}$, and satisfies

$$\lambda_1^\epsilon \geq \frac{9}{10} \log L\,.$$

Comments on Theorem

$$F(x, y) := (2x + L\sin(2\pi x) - y,\, x)$$

- No assumptions made on detailed dynamics of F:
  - Elliptic fixed points and periodic points allowed.
  - Typical length T of sojourn to vicinity of elliptic fixed point:

$$T \approx \epsilon^{-1} = L^{cL^{9/10}}.$$

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- No assumptions made on detailed dynamics of F:
  - Elliptic fixed points and periodic points allowed.
  - Typical length T of sojourn to vicinity of elliptic fixed point:
    $$T \approx \epsilon^{-1} = L^{cL^{9/10}}.$$
- By precluding elliptic periodic points of period ≤ 3, we can allow
  $$\epsilon > L^{-cL^{19/10}}.$$

LE and decay of correlations ($b \le 1$)

$$F(x,y) := (2x + L\sin(2\pi x) - y,\ x), \qquad F_{\omega_i}(x,y) = F(x+\omega_i, y),$$
$$F^n_{\underline{\omega}} = F_{\omega_n} \circ \cdots \circ F_{\omega_1}$$

Note: sinks possible! Need ϵ larger.

Theorem (Joint with JX and LSY; CMP 2017)

Let $b \in (0,1]$. Then there exists $L_0 = L_0(b) > 0$ such that for any $L \ge L_0$ and $\epsilon \ge L^{-9/10}$, we have

- *the top Lyapunov exponent λ_1^ϵ exists almost surely and satisfies $\lambda_1^\epsilon \ge \frac{9}{10}\log L$; and*
- *There exists $K_0 \in \mathbb{N}, \sigma > 0$ such that*

$$\left| \int \phi d(\mu_1 P^n) - \int \phi d(\mu_2 P^n) \right| \le L^{-\sigma(n-K_0)}.$$

for all $\phi \in L^\infty$, μ_1, μ_2 Borel probabilities, $n \ge K_0$.

Comments on Theorem

$$F(x, y) := (2x + L\sin(2\pi x) - y,\ bx)$$

- No assumptions on detailed dynamics of $F-$ sinks could exist!
  - Sinks have basins of size $O(L^{-1})$; perturbations are just large enough **to escape with high probability**

Comments on Theorem

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- No assumptions on detailed dynamics of $F-$ sinks could exist!
  - Sinks have basins of size $O(L^{-1})$; perturbations are just large enough **to escape with high probability**
- Precluding sinks of period ≤ 3 permits us to take $\epsilon \geq L^{-19/10}$ instead.

Additional work:

"Shaking loose" hyperbolicity / expansion for predominantly hyperbolic systems:

- Lian-Stenlund '12 : 1D maps (essentially our model with $k = 0$)
- Ledrappier-Shub-Simo-Wilkinson '03 : Random perturbations of twist maps on sphere
- "à là Furstenberg": typical random cocycles have simple Lyapunov spectrum

Closely related 1D work:

- Katok-Kifer '86: zero noise limits of Misurewicz maps
- Baladi, Benedicks, Maume-Deschamps '00: quenched correlation decay for small random perturbations of Misurewicz maps

Conclusions

- Small random perturbations simplify estimation of Lyapunov exponents
- Methods rely only on **checkable** dynamical properties.
  - Amenable to broad generalization (e.g. higher dimension)
- Not so unnatural from modeling standpoint: the real world is inherently noisy!

Thank you!