Spatio-temporal processes for visits to small balls

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Work in collaboration with Benoît Saussol

New developments in Open Dynamical Systems and their Applications
Banff, March 21th, 2018

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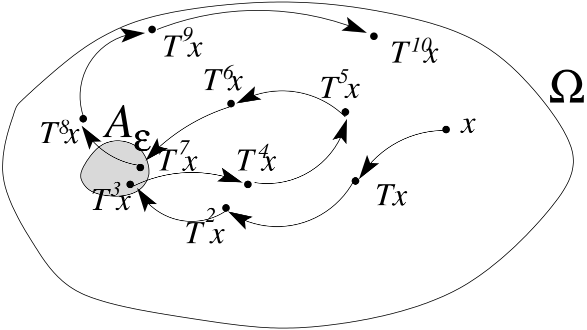

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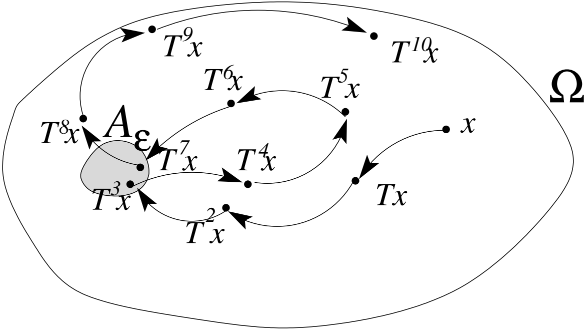

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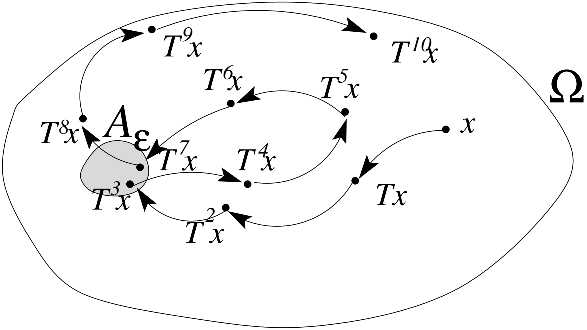

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Here: We study the spatio-temporal point process.

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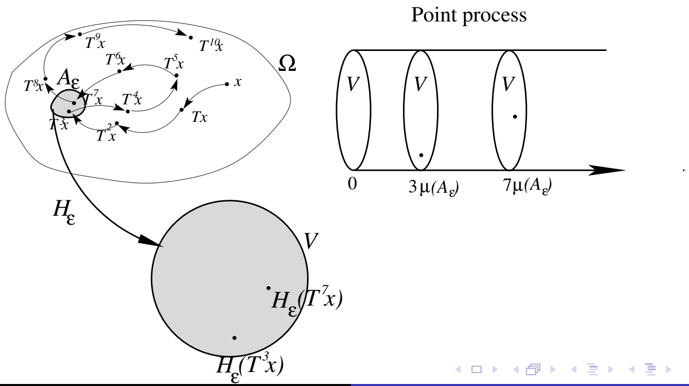

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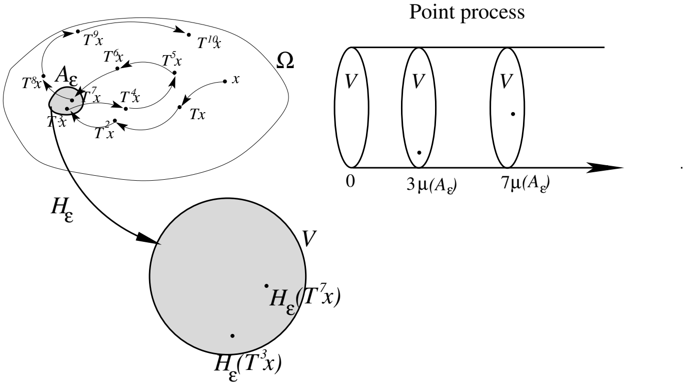

In formulas: $\mathcal{N}_\varepsilon : x \mapsto \sum_{n\geq 1\ :\ T^n(x)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(T^n x))}$,

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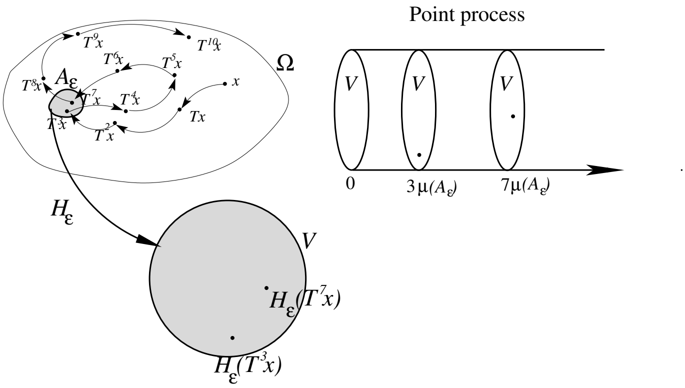

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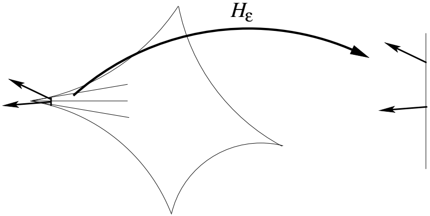
For $B = (a, b) \times W$ with $W \subset V$, $y \in \Omega$

$$\mathcal{N}_\varepsilon(B)(x) = \#\left\{ n \geq 1\ :\ \frac{a}{\mu(A_\varepsilon)} < n < \frac{b}{\mu(A_\varepsilon)},\ H_\varepsilon(T^n(x)) \in W \right\}$$

Motivations

- **Question by D. Szász and I. P. Tóth**: Diamond billiard without cusp



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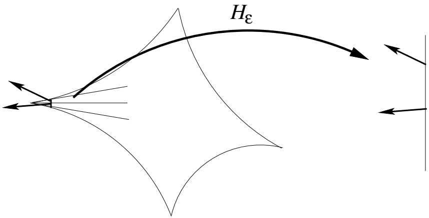

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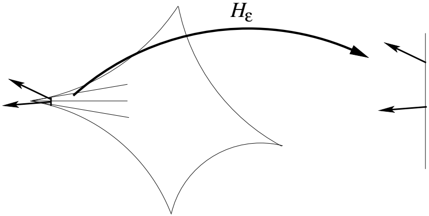

- **Applications to the convergence to Lévy processes** [M. Tyran-Kamińska], [I. Melbourne, R. Zweimüller]
- $A_\varepsilon = B(x_0, \varepsilon)$ with x_0 a generic point or x_0 a hyperbolic periodic point.

Approximation by a Poisson Point Process

- $(\Omega, \mathcal{F}, \mu, T)$, $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \to 0+$, $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$.
  Point process:
  $$\mathcal{N}_\varepsilon := \sum_{n\ :\ T^n(\cdot)\in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(T^n\cdot))},$$
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- Let η be a measure on $(E, \mathcal{E}) = ((0, +\infty) \times V, \mathcal{B}(E))$.
  "$\mathcal{P}_0$ is a $PPP(\eta)$" means:
  - $\mathcal{P}_0$ has values in the set of point measures on $(E, \mathcal{E})$
  - $\forall$ pairwise disjoint $B_1, ..., B_N \in \mathcal{E}$, $\mathcal{P}(B_1), ..., \mathcal{P}(B_N)$ are independent Poisson random variables with respective expectations $\eta(B_1), ..., \eta(B_N)$.

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- "$\mathcal{N}_\varepsilon \Longrightarrow_{\varepsilon \to 0} \mathcal{P}_0 : \; PPP(\eta)$" implies:
  $\forall$ pairwise disjoint $B_1, ..., B_N \in \mathcal{B}((0, +\infty) \times V)$ s.t. $\eta(\partial B_i) = 0$,

$$(\mathcal{N}_\varepsilon(B_1), ..., \mathcal{N}_\varepsilon(B_N)) \Longrightarrow_{\varepsilon \to 0} (\mathcal{P}_0(B_1), ..., \mathcal{P}_0(B_N)),$$

  in distribution wrt any probability measure $\mathbb{P} \ll \mu$.

Approximation by a PPP: a general result

- $(\Omega, \mathcal{F}, \mu, T)$, $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \to 0+$, $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$ and

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- **Theorem** [Pène,Saussol 2018]
Let $\mathcal{W}$ be a family of relatively compact open subsets of V, stable by $\cup$ and $\cap$ and generating $\mathcal{B}(V)$ such that $\forall$ finite $\mathcal{W}_0 \subset \mathcal{W}$,

$$\sup_{A\in H_\varepsilon^{-1}\mathcal{W}_0, B\in\bigcup_{n\geq 1}\sigma(T^{-n}H_\varepsilon^{-1}\mathcal{W}_0)} |\mu(B\cap A) - \mu(B)\mu(A)| = o(\mu(A_\varepsilon)).$$

If there exists a measure m on V s.t. $\forall F \in \mathcal{W}$: $m(\partial F) = 0$ and $\mu\left(H_\varepsilon^{-1}(F)|A_\varepsilon\right) \to m(F)$, then $\mathcal{N}_\varepsilon \Longrightarrow_{\varepsilon\to 0} PPP(\lambda \times m)$.
vague convergence is enough! How does $\mathcal{W}$ look like?

About our technical assumption

- **Hypothesis.** *There exists $\mathcal{W}$ a family of relatively compact open subsets of V, stable by $\cup$ and $\cap$ and generating $\mathcal{B}(V)$ such that:* $\forall F \in \mathcal{W},\ m(\partial F) = 0$ *and* $\forall$ *finite* $\mathcal{W}_0 \subset \mathcal{W}$,

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- **Idea**: short returns are rare + decorrelation.

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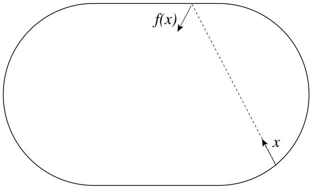
- **Idea**: short returns are rare + decorrelation.
- **Hypothesis true** if V is open and if (Ω, μ, T) is an (expanding/)hyperbolic dynamical systems modeled by a Young tower (Δ, T, ν) (context of [J.F. Alves, D. Azevedo]) with $\alpha, \beta > 0$ such that, on the base of the tower:

$$diam(T^k(\gamma^s)) + diam(T^{-k}(\gamma^u)) \leq Ck^{-\alpha}, \ Leb_{\gamma^u}(R > n) \leq Cn^{-1-\beta}$$

(Sinai+Diamond billiard: any $\alpha, \beta > 0$, Bunimovich Stadium billiard: $\alpha = \beta = 1$) AND if $\exists p_\varepsilon \ll \mu(A_\varepsilon)^{-1}$ s.t.

$$\mu(\tau_{A_\varepsilon} \leq p_\varepsilon | A_\varepsilon) = o(1) \quad \text{and} \quad \mu\left((\partial A_\varepsilon)^{[p_\varepsilon^{-\alpha}]}\right) = o(\mu(A_\varepsilon)).$$

Billiard in the stadium



Q: billiard domain.

Billiard map $(M = \partial Q \times S^1, \mu, T)$, $\frac{d\mu}{d\lambda}(q, \vec{v}) \sim \langle \vec{n}_q^{inward}(\partial Q), \vec{v} \rangle^+$.

Theorem (Billiard in the stadium).[Pène,Saussol2015+2018]

For μ-a.e. $x_0 = (q_0, \vec{v}_0) \in M$, if $A_\varepsilon = B(x_0, \varepsilon)$ and $H_\varepsilon(q, \vec{v}) = (q - q_0, \angle(\vec{v}_0, \vec{v}))/\varepsilon$, $\mathcal{N}_\varepsilon \Longrightarrow_{\varepsilon \to 0}$ Poisson process$(\lambda_3/4)$, with λ_3 the Lebesgue measure on $(0, +\infty) \times (-1, 1)^2$.

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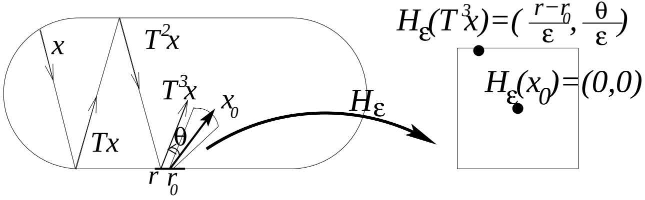

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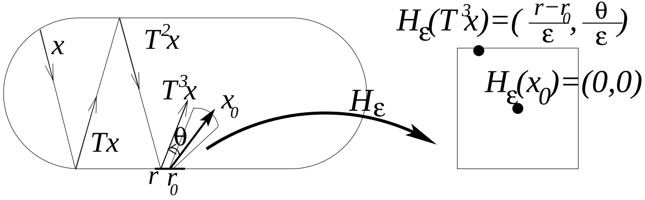

Proof.

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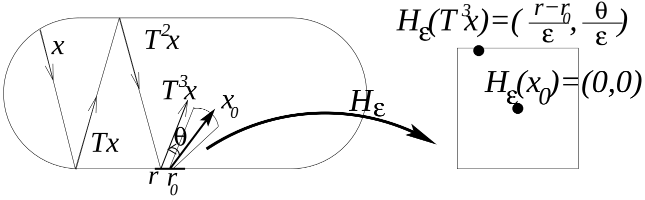

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- A careful reading of [Chernov,Markarian2006] leads to: $\text{diam}(T^n \gamma^s) \le Cn^{-\alpha}$, $\text{diam}(T^{-n}\gamma^u) \le Cn^{-\alpha}$, $\alpha = 1$

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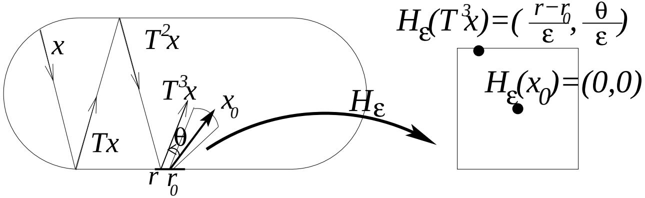

Proof.

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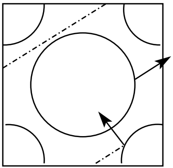
- $\operatorname{diam}(T^n \gamma^s) \le C n^{-\alpha}$, $\operatorname{diam}(T^{-n} \gamma^u) \le C n^{-\alpha}$, $\alpha = 1$
- $p_\varepsilon = \varepsilon^{-\sigma}$ with $\sigma \in (1, 2)$ satisfies: $p_\varepsilon \ll \mu(A_\varepsilon)^{-1} \sim c\varepsilon^{-2}$ and

$$\mu((\partial A_\varepsilon)^{[p_\varepsilon^{-\alpha}]}) \approx \varepsilon p_\varepsilon^{-\alpha} = o(\varepsilon^2) = o(\mu(A_\varepsilon)).$$

A results for the Sinai billiard flow

- Billiard domain: $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} O_i$, finite horizon.



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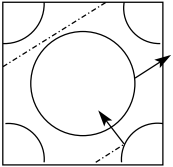

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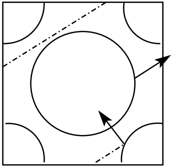

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- Billiard flow $(\mathcal{M} = Q \times S^1, \nu, (Y_t)_t)$, $\nu = Leb(\cdot|\mathcal{M})$. Let $q_0 \in Q$.

$$\mathfrak{N}_{\varepsilon,q_0} = \sum_{t\,:\, Y_t \text{ enters } B(q_0,\varepsilon)\times S^1} \delta_{\left(\frac{2\pi\varepsilon t}{Area(Q)},(\varepsilon^{-1}(\Pi_Q(Y_t(y))-q_0)),\Pi_V(Y_t(y))\right)}.$$

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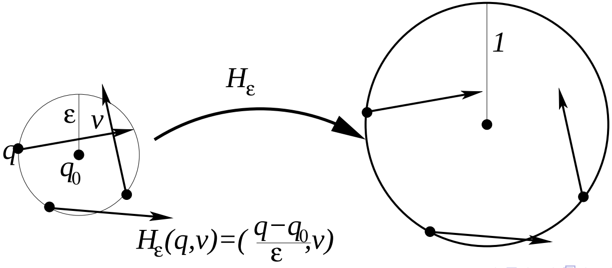
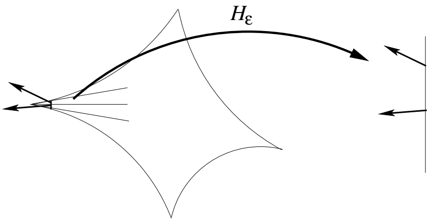
Theorem *For Lebesgue-almost every $q_0 \in Q$,*
$\mathfrak{N}_{\varepsilon,q_0} \Rightarrow PPP(\frac{1}{4\pi}\langle\vec{n}^{inward}_{S^1}(q),\vec{v}\rangle^+\mathbf{1}_{[0,+\infty)\times S^1\times S^1}\,dtdqd\vec{v})$,



Diamond billiard without cusp

- Put a vertical barrier of length ε close to the left corner $(0,0)$ (bisector is here horizontal).



$H_\varepsilon\left((q_1,q_2),\vec{v}\right)=\left(\frac{q_2}{\varepsilon},\vec{v}\right)$,
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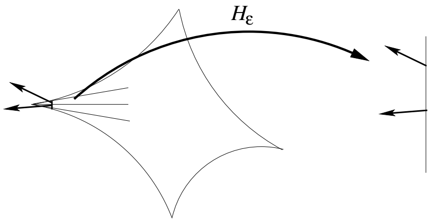

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- **Theorem.**

$$\sum_{t>0:\,Y_t\in\mathcal{A}_\varepsilon}\delta_{\left(\frac{\varepsilon t}{\pi Area(Q)},H_\varepsilon(Y_t(\cdot))\right)}\Longrightarrow PPP\left(|\cos(\varphi)|dtdqd\varphi\right)$$

Application to hyperbolic fixed points

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- Notations: $A_\varepsilon:=B(x_0,\varepsilon)\setminus\bigcup_{j=1}^{q_0}T^{-j}(B(x_0,\varepsilon))$
  (q_0 so that $A_\varepsilon\cap T^{-n}(A_\varepsilon)=\emptyset$, $\forall n=1,...,-a\log\varepsilon$)
  $H_\varepsilon: B_\Omega(x_0,\varepsilon)\setminus\{x_0\}\to B(0,1)\setminus\{0\}$, $H_\varepsilon(x)=\varepsilon^{-1}\exp^{-1}(x)$

$$\mathcal{N}_\varepsilon:=\sum_{n\,:\,T^n(\cdot)\in A_\varepsilon}\delta_{(n\mu(A_\varepsilon),H_\varepsilon(T^n\cdot))},\quad \mu_\varepsilon:=\mu\left(H_\varepsilon^{-1}(\cdot)|A_\varepsilon\right),$$

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- Conclusion: $\forall f: M_p(E)\to\mathbb{R}$ continuous bounded, $\forall\mathbb{P}\ll\mu$,
  $\mathbb{E}\left[f(\mathcal{N}_\varepsilon)\right]-\mathbb{E}\left[f(PPP(\lambda\times\mu_\varepsilon)\right]\to 0,\quad \mathbb{E}\left[f(\tilde{\mathcal{N}}_\varepsilon)\right]-\mathbb{E}\left[f\left(\Psi\left(PPP(\lambda\times\right.\right.\right.$
  with $\Psi\left(\sum_n\delta_{(t_n,y_n)}\right) := \sum_n\sum_{k=0}^{\ell_{y_n}}\delta_{(t_n,DT_{x_0}^{-k}(y_n))}$ and
  $\ell_y := \inf\left\{k\geq 0\ :\ DT_{x_0}^{-k}(y)\in B(0,1)\setminus\bigcup_{j=0}^{q_0}DT_{x_0}^{j}B(0,1)\right\}$

A partial bibliography on quantitative recurrence

- cylinder sets: [Hirata1993], [Hirata,Saussol,Vaienti1999], [Bruin,Vaienti2003], [Abadi,Vergne2008], [Haydn,Vaienti2004].
- Uniformly expanding maps [Collet,Galves1995].
- Non-uniformly expanding maps (intermittent maps) [Collet,Galves1993], [Bruin,Saussol2003], [Bruin,Vaienti2003], [Collet2001], [Freitas,Freitas,Todd2010], [Holland,Nicol,Török].
- some partially hyperbolic systems [Dolgopyat]
- Sinai billiard, Axiom A attractors with one-dimensional unstable manifolds [Collet,Chazottes2013].
- weakly hyperbolic systems [Haydn,Wasilewska2014], [Pène,Saussol2015]
- Bunimovich billiard: [Freitas,Haydn,Nicol2014], [Pène,Saussol2015].
- [Carvalho,Moreira Freitas,Freitas,Holland,Nicol2015] periodic points