Thermodynamics of the Katok Map

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New Developments in Open Dynamical Systems
and Their Applications
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Classical Thermodynamic Formalism (Sinai, Ruelle, Bowen)

X a compact metric space, $f : X \to X$ a continuous map of finite topological entropy, φ a continuous function (potential) on X, $M(f)$ the space of all f-invariant Borel probability ergodic measures on X. $\mu_\varphi \in M(f)$ is an equilibrium measure if

$$P(\varphi) := \sup_{\mu \in M(f)} \{h_\mu(f) + \int_X \varphi \, d\mu\} = h_{\mu_\varphi}(f) + \int_X \varphi \, d\mu_\varphi.$$

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Existence and uniqueness of equilibrium measures are well known in the case when f is diffeomorphisms with a locally maximal hyperbolic set Λ ($f|\Lambda$ is assumed to be topologically transitive) and φ is Hölder continuous.

Ergodic Properties of Equilibrium Measures μ_φ

1. μ_φ is ergodic and if $f|\Lambda$ is topologically mixing, then μ_φ is a Bernoulli measure.
2. μ_φ has exponential decay of correlations with respect to the class of Hölder continuous observables on X that is for every Hölder continuous h_1 and h_2 the correlation function
$$\mathrm{Cor}_n(h_1, h_2) := \left| \int h_1(f^n(x))h_2(x)d\mu_\varphi - \int h_1(x)d\mu_\varphi \int h_2(x)d\mu_\varphi \right|$$
satisfies $\mathrm{Cor}_n(h_1, h_2) \le Ce^{-an}$, where $C = C(h_1, h_2) > 0$ and $a > 0$ are constants.
3. μ_φ satisfies the Central Limit Theorem (CLT) with respect to the class of Hölder continuous observables on X that is for every Hölder continuous h with $\int h\, d\mu_\varphi = 0$ the sum
$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} h(f^i(x))$$
converges in law to a normal distribution $N(0, \sigma)$.

The Geometric Potential

Of special interest is the geometric t-potential: a family of potential functions $\phi_t(x) := -t \log |\mathrm{Jac}(df|E^u(x))|$ for $t \in \mathbb{R}$. Since the subspaces $E^u(x)$ depend Hölder continuously on x, the potential ϕ_t is Hölder continuous for each t and hence, admits a unique equilibrium measure μ_t. Observe that μ_0 is the unique measure of maximal entropy. The pressure function $P(t) := P(\phi_t)$ is convex, decreasing, and real analytic in t.

Away from uniform hyperbolicity there are classes of dynamical systems for which thermodynamical formalism is rather well understood. In particular, existence and uniqueness of equilibrium measures for the geometric t-potential is established for some intervals in t and ergodic properties of these measures are known up to the decay of correlations. Moreover, phase transitions where the pressure function is non-differentiable are found.

Away from uniform hyperbolicity there are classes of dynamical systems for which thermodynamical formalism is rather well understood. In particular, existence and uniqueness of equilibrium measures for the geometric t-potential is established for some intervals in t and ergodic properties of these measures are known up to the decay of correlations. Moreover, phase transitions where the pressure function is non-differentiable are found.
These include (the list is far from being complete):

- One-dimensional maps: unimodal and multimodal maps (Keller–Bruin, Todd, Dobbs, Iommi, Senti–Pesin, etc.); maps with an indifferent fixed point (Hu, Young, Sarig, etc.).
- Polynomial and rational maps (Przytycki–Letellier, Makarov–Smirnov, etc.).
- (Piecewise) non-uniformly expanding maps (Buzzi, Sarig, Alves–Luzzatto–Pinheiro, Oliveira, Varandas, Viana).

However, there are only few known examples of smooth dynamical systems in dimension ≥ 2 for which there is a sufficiently complete description of their thermodynamics with respect to the geometric t-potential. These include:

- the Hénon map at the first bifurcation for t in an interval $(-t_0, 1)$ with $t_0 > 0$ (Senti–Takahashi).
- Mañé and Bonatti-Viana maps for t in an interval $(t_0, t_1) \supset [0, 1]$ (Climenhaga–Fisher–Thompson).
- Geodesic flows on compact surfaces of non-positive curvature for t in the interval $(-\infty, 1)$ (Burns–Climenhaga–Fisher–Thompson).
- The slow-down of the Smale-Williams solenoid for t in an interval $(-t_0, 1)$ with $t_0 > 0$ (Zelerowicz).
- the Katok map (the slow-down of a hyperbolic automorphism of the 2-torus) for t in an interval $(-t_0, 1)$ with arbitrary large $t_0 > 0$ (P.–Senti–Zheng).

The Katok Map

Consider the automorphism A of the two-dimensional torus $\mathbb{T}^2$ given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and choose a function $\psi : [0,1] \mapsto [0,1]$ satisfying:

1. ψ is C^∞ everywhere except at the origin;
2. $\psi(u) = 1$ for $u \geq r_0$ and some $0 < r_0 < 1$;
3. $\psi'(u) > 0$ for every $0 < u < r_0$;
4. $\psi(u) = (u/r_0)^\alpha$ for $0 \leq u \leq \frac{r_0}{2}$ and some $0 < \alpha < 1$.

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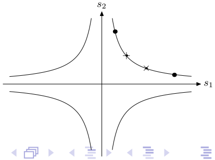
Let $D_r = \{(s_1, s_2) : {s_1}^2 + {s_2}^2 \leq r^2\}$ where (s_1, s_2) is the coordinate system obtained from the eigendirections of A. Let $\lambda > 1$ be an eigenvalue of A and $r_1 = (\log \lambda) r_0$, so that

$$D_{r_0} \subset \mathrm{Int} A(D_{r_1}) \cap \mathrm{Int} A^{-1}(D_{r_1}).$$

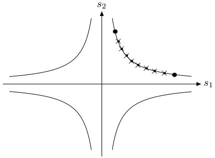
The map A is the time-1 map of the flow generated by the system of ODE:

$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda.$$



We slow down trajectories by perturbing the flow in D_{r_0} as follows

$$\dot{s}_1 = s_1 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$
$$\dot{s}_2 = -s_2 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$



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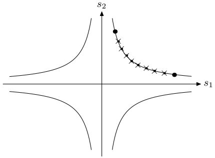


This system of equations generates a local flow and let g be the time-one map of this flow. g is of class C^∞ except at the origin and it coincides with A in some neighborhood of the boundary ∂D_{r_1}. Therefore, the map

$$G(x) = \begin{cases} A(x) & \text{if } x \in \mathbb{T}^2 \setminus D_{r_0}, \\ g(x) & \text{if } x \in D_{r_0} \end{cases}$$

defines a homeomorphism of the torus $\mathbb{T}^2$, which is a C^∞ diffeomorphism everywhere except at the origin.

Since $0 < \alpha < 1$, we obtain that

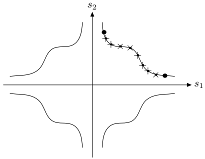
$$\int_0^1 \frac{du}{\psi(u)} < \infty.$$

This implies that the map G preserves the probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$, where m is the area and the density κ is a positive C^∞ function that is infinite at 0 and is defined by

$$\kappa(s_1, s_2) := \begin{cases} (\psi({s_1}^2 + {s_2}^2))^{-1} & \text{if } (s_1, s_2) \in D_{r_0}, \\ 1 & \text{otherwise} \end{cases}$$

and $\kappa_0 := \int_{\mathbb{T}^2} \kappa\, dm$.

The map g can be further perturbed to a map F that preserves area and has nonzero Lyapunov exponents almost everywhere.



This can be done by a coordinate change ϕ in $\mathbb{T}^2$. Define ϕ in D_{r_0} by the formula

$$\phi(s_1, s_2) := \frac{1}{\sqrt{\kappa_0({s_1}^2 + {s_2}^2)}} \left(\int_0^{{s_1}^2+{s_2}^2} \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2)$$

and set $\phi = \mathrm{Id}$ in $\mathbb{T}^2 \setminus D_{r_0}$. Clearly, ϕ is a homeomorphism and is a C^∞ diffeomorphism outside the origin. One can show that ϕ transfers the measure ν into the area and that $F = \phi \circ G \circ \phi^{-1}$ is a $C^{1+\kappa}$ diffeomorphism where $\kappa = \frac{\alpha}{1-\alpha}$. It is called the Katok map (Katok, 1979).

Properties of the Katok Map

1. F is topologically conjugate to A by a homeomorphism H.
2. There exist two continuous, uniformly transverse, invariant one-dimensional stable $E^s(x)$ and unstable $E^u(x)$ distributions.
3. For almost every $x \in \mathbb{T}^2$ the Lyapunov exponent $\chi(x, v) > 0$ for $v \in E^u(x)$ and $\chi(x, v) < 0$ for $v \in E^s(x)$; the Lyapunov exponents at the origin are equal to zero;
4. F has two continuous, uniformly transverse, invariant one-dimensional foliations with smooth leaves. They are stable W^s and unstable W^u foliations for F and are the images under H of the stable and unstable foliations for A respectively.
5. F is ergodic and in fact, is isomorphic to a Bernoulli diffeomorphism.

Main Theorem (P.–Senti–Zhang)

Consider the geometric potential $\varphi_{t,F} = -t \log |\mathrm{Jac}(dF|E^u(x))|$. Then for arbitrary large $t_0 > 0$ there is $r_0 > 0$ such that

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1. For every $-t_0 < t < 1$ there exists a unique equilibrium measure μ_t associated to the geometric potential $\varphi_{t,F}$.
2. There are two equilibrium measures for $\varphi_{1,F}$ – the area m and the Dirac measure at the origin.
3. For every $t > 1$ there exists a unique equilibrium measure μ_t – the Dirac measure at the origin and $P(t) = 0$.

Main Theorem (continued)

The measure μ_t has the following ergodic properties: for every $-t_0 < t < 1$

- μ_t has exponential decay of correlations with respect to the class of Hölder continuous observables on $\mathbb{T}^2$.
- μ_t satisfies the CLT with respect to the class of Hölder continuous observables on $\mathbb{T}^2$.
- (Shahidi–Zelerovich) μ_t is a Bernoulli measure.
- (P.–Senti–Shahidi) $\mu_1 = m$ has polynomial decay of correlations with respect to the class of Hölder continuous observables on $\mathbb{T}^2$ and admits a polynomial lower bound for correlations with respect to a nested sequence of subsets $\{M_k\}$ that exhausts $\mathbb{T}^2$ and the class of Hölder continuous observables h for which there is $k = k(h)$ such that $\text{supp}(h) \subset M_k$.

Some Open Problems

1. Is the function $t \to \mu_t$ continuous (smooth) in t on $(-t_0, 1)$?
2. Does $\mu_t \to m$ as $t \to 1$?
3. Is there $t_1 \geq t_0$ such that the pressure function $P(t)$ is not differentiable at $-t_1$?

Proof of Main Theorem

Consider a finite Markov partition $\tilde{\mathcal{P}}$ for the automorphism A and let $\tilde{P} \in \tilde{\mathcal{P}}$ be a partition element which lies "away" from the origin. The set $\tilde{P}$ is a rectangle with direct product structure generated by full length stable $\tilde{\gamma}^s(x)$ and unstable $\tilde{\gamma}^u(x)$ curves (i.e., segments of stable and unstable lines). Moreover, given $\tilde{\varepsilon} > 0$, we can choose the partition such that its diameter $\leq \tilde{\varepsilon}$.

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For $x \in \tilde{P}$ let $\tau(x)$ be the first return time of x to Int $\tilde{P}$. For all x with $\tau(x) < \infty$, the connected component of the level set $\tau = \tau(x)$ containing x is an s-set

$$\tilde{\Lambda}^s(x) = \bigcup_{y \in \tilde{U}^u(x) \setminus \tilde{A}^u(x)} \tilde{\gamma}^s(y),$$

where $\tilde{U}^u(x) \subseteq \tilde{\gamma}^u(x)$ is an interval containing x and $\tilde{A}^u(x) \subset \tilde{U}^u(x)$ is the set of points which either lie on the boundary of the Markov partition or never return to the set $\tilde{P}$. Note that $\tilde{A}^u(x)$ has zero Lebesgue measure in $\tilde{\gamma}^s(x)$.

The sets $\tilde{\Lambda}^u(x) = F^{\tau(x)}(\tilde{\Lambda}^s(x))$ are u-sets. Thus, we obtain a countable collection of disjoint s-sets $\tilde{\Lambda}_i^s$ and the corresponding collection of disjoint u-sets $\tilde{\Lambda}_i^u = F^{\tau_i}(\tilde{\Lambda}_i^s)$ where τ_i is the first return of any point $x \in \tilde{\Lambda}_i^s$ to $\tilde{\mathcal{P}}$.

Applying the conjugacy map H to the Markov partition $\tilde{\mathcal{P}}$, we obtain a Markov partition $\mathcal{P}$ for F of diameter ε (and $\varepsilon \to 0$ as $\tilde{\varepsilon} \to 0$) and we consider the element $P = H(\tilde{P})$. The set P is a rectangle with direct product structure generated by full length stable $\gamma^s(x)$ and unstable $\gamma^u(x)$ curves where $\gamma^{s,u}(x) = H(\tilde{\gamma}^{s,u}(x))$.

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The sets $\{\Lambda_i^{s,u} = H(\tilde{\Lambda}_i^{s,u})\}$ are respectively the s- and u-sets, they are disjoint, and the number τ_i is the first return of any point $x \in \Lambda_i^s$ to $\mathcal{P}$. Observe that

$$\Lambda_i^s = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y),$$

where $U^u(x) \subseteq \gamma^u(x)$ is an open set containing x and $A^u(x) \subset U^u(x)$ is the set of points which either lie on the boundary of the Markov partition or never return to the set P. Note that $A^u(x)$ has zero Lebesgue measure in $\gamma^s(x)$.

The set

$$\Lambda = \left(\bigcup_i \Lambda_i^s\right) \cap \left(\bigcup_i \Lambda_i^u\right)$$

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has direct product structure generated by full length stable $\gamma^s(x)$ and unstable $\gamma^u(x)$ curves. Thus we obtain a representation of the Katok map F as a tower with base Λ and the height function τ which is constant τ_i on each partition element Λ_i^s.

We have for every $x \in \Lambda_i^s$:

1. Invariance property:

$$f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)), \; f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x));$$

2. Markov property:

$$f^{\tau_i}(\gamma^u(x) \cap \Lambda_i^s) = \gamma^u(f^{\tau_i}(x)) \cap \Lambda$$

(follows from the Markov property of the partition $\mathcal{P}$);

3. Integrable tail of the height function:

$$\sum_{i=1}^{\infty} \tau_i m_{\gamma^u(x)}(\Lambda_i^s) < \infty,$$

where m_{γ^u} is the one-dimensional Lebesgue measure on $\gamma^u(x)$ (follows from the facts that τ is the first return time to Λ and f preserves area).

Define the induced map $\mathcal{F} : \bigcup \Lambda_i^s \to \Lambda$ by $F|\Lambda_i^s := f^{\tau_i}|\Lambda_i^s$. We require that there exists $0 < a < 1$ such that

(a) $d(\mathcal{F}(x), \mathcal{F}(y)) \leq a\, d(x, y)$ for $x \in \Lambda_i^s$ and $y \in \gamma^s(x)$;

(b) $d(x, y) \leq a\, d(\mathcal{F}(x), \mathcal{F}(y))$ for $x \in \Lambda_i^s$ and $y \in \gamma^u(x) \cap \Lambda_i^s$.

(c) $\mathcal{F}$ has the bounded distortion property.

This means that the induced map $\mathcal{F}$ is uniformly hyperbolic on the non-invariant set Λ.

Diffeomorphisms admitting a tower representation satisfying the above properties are called Young diffeomorphisms.

Establishing Properties (a), (b), and (c) is the most difficult technical part of the proof and requires a deeper understanding of the properties of the Katok map.

Additional Properties of the Katok Map

For $x \in \Lambda$ with $\tau(x) < \infty$ define the itinerary of x,

$$0 = n_0 < n_1 < \cdots < n_{2k} < n_{2k+1} = \tau(x), \; n_i = n_i(x), \; k = k(x)$$

and $F^j(x) \in D_{r_0}$ if and only if $n_{2\ell-1} \le j < n_{2\ell}$ for some $1 \le \ell \le k$.

(1) Then $n_{2\ell+1} - n_{2\ell} \ge Q$ uniformly in x, where Q can be chosen arbitrarily large if the number r_0 is small.

Additional Properties of the Katok Map

(2) Given $x \in \Lambda$, $y \in \gamma^s(x)$, denote $x_n = F^n(x)$ and $y_n = F^n(y)$. Note that y_n lies on the stable curve through x_n. Then there is $C_4 > 0$ such that

$$d(x_{n_{2l}}, y_{n_{2l}}) \leq Cd(x_{n_{2l-1}}, y_{n_{2l-1}}).$$

(3) There is $0 < b < 1$ such that

$$d(x_{n_{2l}}, y_{n_{2l}}) \leq b^Q d(x_{n_{2l+1}}, y_{n_{2l+1}}).$$

Statements (1)–(3) imply the desired properties of the induced map $\mathcal{F}$.

The following general result describes existence, uniqueness and ergodic properties of Young diffeomorphisms for the geometric potential $\varphi_t(x) := -t \log |df|E^u(x))|$. We only consider the case when the height of the tower is the first return time to the base. The result can be extended to the general case.

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We say that the tower satisfies the arithmetic condition if the greatest common denominator of the set of integers $\{\tau_i\}$ is one.

We say that the tower satisfies the Melbourne–Terhesiu condition if for every $i \geq 0$, $x, y \in \Lambda_i^s$, $0 \leq j \leq \tau_i$ we have

$$d(f^j(x), f^j(y)) \leq K \max\{d(x, y), d(F(x), F(y))\}.$$

Denote by $S_n := \text{Card}\ \{\Lambda_i^s \colon \tau_i = n\}$.

Theorem (P., Senti, Zhang)

1. *Assume that* $S_n \le e^{hn}$ *with* $0 < h < -\int \varphi_1 \, d\mu_1$. *Then there is* $t_0 > 0$ *such that for every* $-t_0 < t < 1$ *there exists a unique equilibrium measure* μ_t *for the potential* φ_t.
2. *Assume the tower satisfies the arithmetic condition and the Melbourne–Terhesiu condition. Then* μ_t *has exponential decay of correlations, and satisfies the CLT with respect to the class of Hölder continuous functions on* M.
3. *(Shahidi–Zelerowicz)* μ_t *is a Bernoulli measure and the pressure function* $P(t)$ *is real analytic in* $t \in (-t_0, 1)$.

Verifying the Exponential Bound on S_n

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Hence, it suffices to estimate the number of sets Λ_i^s with a given i. This number is less than the number of periodic orbits of A that originate in $\tilde{P}$ and have minimal period τ_i. Using the symbolic representation of A as a subshift of finite type, one can see that the latter equals the number of symbolic words of length τ_i for which the symbol $\tilde{P}$ occurs only as the first and last symbol (but nowhere in between). The number of such words grows exponentially with exponent $h < h_{\text{top}}(A)$.

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Observe that $h < -\int \varphi_1 \, d\mu_1$. Indeed, this holds for the automorphism A, and hence, for F provided r_0 is sufficiently small.

Completing the Proof of Main Theorem

Note that $\mu_t(U) > 0$ for every open set $U \subset P$. Let us choose another element $\hat{P}$ of the Markov partition which is "far away" from the origin. Repeating the above argument there exists a unique ergodic equilibrium measure $\hat{\mu}_t$ for the geometric potential among all measures μ for which $\mu(\hat{P}) > 0$ and $\hat{\mu}_t(\hat{U}) > 0$ for every open set $\hat{U} \subset \hat{P}$. Since the map F is topologically transitive, for any open sets $U \subset P$ and $\hat{U} \subset \hat{P}$ there exists k such that $F^k(U) \cap \hat{U} \neq \emptyset$. Therefore, $\mu_t = \hat{\mu}_t$.

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If the number r_0 in the construction of the Katok map is sufficiently small, the union of partition elements that are "far away" from the origin form a closed set whose complement is a neighborhood of the origin. The desired result now follows by observing that the only measure which does not charge any element of the Markov partition that lies outside this neighborhood is the Dirac measure at the origin.

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Finally, we let $t_0 = \min\, t_0(P)$.

Happy Birthday Leonya!