Deterministic particle approximations for transport models with nonlinear mobility

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Transport/Diffusion models with nonlinear mobility

Continuity equations with local/nonlocal transport and/or diffusion:

$$\partial_t \rho + \mathrm{div}(\rho V[\rho, \nabla\rho]) = 0\,, \qquad \rho = \rho(t,x), \quad x \in \mathbb{R}^d\,,\ t \geq 0.$$

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In relevant examples, the *speed* $|V[\rho, \nabla\rho]|$ is *cut-off at high densities* no matter what the direction is: **prevention of overcrowding:**

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Typical law for $v(\rho)$

Let $\rho_{\max} > 0$ be a maximal density. Then,

$$v : [0, \rho_{\max}] \to [0, v_{max}], \qquad v \text{ decreasing, and } v(\rho_{\max}) = 0.$$

Examples in biology

Aggregation/Swarming phenomena

Avoiding blow-up of the density when aggregation phenomena dominate, with the goal of detecting *pattern-formation* for large times rather than concentration to Dirac masses:

$$\partial_t \rho + \mathrm{div}(\rho v(\rho) \nabla (a(\rho) + W * \rho)) = 0,$$

with $a(\rho)$ a nonlinear diffusion function and $W = W(x)$ an aggregative kernel.

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Modified chemotaxis modelling

To prevent concentration in the Keller-Segel system:

$$\rho_t = D_\rho \Delta \rho - \chi \mathrm{div}(\rho(\rho_{\max} - \rho) \nabla c)$$
$$\varepsilon c_t = D_c \Delta c + \alpha \rho - \beta c.$$

Examples in real-world applications

Traffic flow: extended LRW equation

$d = 1$, vehicles moving in the same direction, external potential $V = V(x)$ describing possible heterogeneities on the road:

$$\partial_t \rho + \partial_x(\rho v(\rho) V'(x)) = 0.$$

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Pedestrian motion

$d = 1, 2$, pedestrians moving with speed $v(\rho)$ and direction $\nabla\varphi/|\nabla\varphi|$, where φ is determined nonlocally from the overall density. Examples:

- (Hughes) $|\nabla\varphi| = c(\rho)$, with cost function $c(\rho)$ increasing with the density.
- (Colombo et al.) $\rho_t + \mathrm{div}\left(\rho v(\rho)(\nu(x) - \varepsilon \frac{\nabla\eta*\rho}{\sqrt{1+|\nabla\eta*\rho|^2}})\right) = 0.$

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Other applications include:

- Phase segregation with long range interactions
- Simplified models for Fermi-Dirac condensates

Comparison with other approaches preventing high densities (Katy Craig's talk).

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Main difficulty

The velocity field is typically not continuous w.r.t. tight convergence, because of the local dependency on ρ.

Related literature

Case v constant (linear mobility):

- Nonlocal interaction equations with regular kernels: Dobrushin (1979)
- Nonlocal interaction equations (with singular kernels): Carrillo et al.
- Deterministic diffusion (linear case): Russo (1990)
- Nonlinear diffusion: Gosse-Toscani (2006)
- Multidimensional deterministic diffusion via gamma convergence: Carrillo, Craig, Patacchini...
- Diffusion and nonlocal interactions: Matthes et al.

Case v non-constant (nonlinear mobility):

- General setting with velocity field continuous w.r.t. measure topology: Piccoli-Rossi 2013

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A discrete lagrangian description

General continuum model:

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Discrete counterpart with N particles $x_1, \dots, x_N$:

$$\dot{x}_i(t) = v(R_i(t))\Big(\nabla V(x_i(t)) + (\nabla W * \tilde{\rho}^N(x_i(t)))\Big),$$

with

- R_i a suitable reconstruction of the density around the particle x_i,
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Fact 1: with $d = 1$ the reconstruction of the density is much easier, and one-dimensional cases are relevant to some of our target applications (e.g. traffic flow).
Fact 2: the nonlocal part can be discretized as

$$\tilde{\rho}^N(t) = \frac{M}{N}\sum_i \delta_{x_i(t)}, \qquad \nabla W * \tilde{\rho}^N(x_i(t)) = \frac{M}{N}\sum_i \nabla W(x_i(t) - x_j(t)),$$

(where M is the total mass).

Scalar conservation laws: the result[1]

The unique entropy solution to the $d = 1$ conservation law

$$\rho_t + (\rho v(\rho))_x = 0,$$

with given $\rho(t = 0) \in L^\infty$, compactly supported, and with mass M, is approximated in L^1 (strongly) as $N \to +\infty$ by the discrete density

$$\rho^N(t, x) = \sum_{i=0}^{N-1} R_i(t)\chi_{[x_i(t), x_{i+1}(t))}\,, \qquad R_i(t) = \frac{\ell_N}{x_{i+1}(t) - x_i(t)}\,, \quad \ell_N = M/N,$$

where x_i, $i = 1, \ldots, N$ solve

$$\dot{x}_N(t) = v(0)$$
$$\dot{x}_i(t) = v(R_i(t))\,, \quad i = 0, \ldots, N-1,$$

with initial condition $\bar{x}_0, \ldots, \bar{x}_N$ such that

$$\int_{\bar{x}_i}^{\bar{x}_{i+1}} \rho(t = 0, x)dx = \ell_N.$$

[1] DF-Rosini, ARMA 2015

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- The scheme leads to the unique *entropy solution*. Surprising, since the scheme does not display the shock structure of a conservation law (kinetic velocity $\neq$ characteristic velocity).
- Approach can be extended to
    - Dirichlet boundary conditions,
    - Second order traffic models,
    - (Small BV-norm solutions to the) Hughes model for pedestrians.

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The continuum nonlocal model

We study[2]

$$\boxed{\begin{aligned} \partial_t \rho - \partial_x(\rho v(\rho) K' * \rho) = 0 \\ \rho(t=0) = \bar{\rho} \in L^\infty_c(\mathbb{R}\,;\,[0,\rho_{\max}]) \cap BV(\mathbb{R}) \end{aligned}} \tag{1a}$$

Assumptions on K

$K \in C^2(\mathbb{R})$, $K(-x) = K(x)$, $K' > 0$ on $(0,+\infty)$, $K'' \in \mathrm{Lip}_{loc}(\mathbb{R})$.

Assumptions on v

$v \in C^1([0,+\infty))$, v decreasing on $[0,\rho_{\max}]$, $v \equiv 0$ on $[\rho_{\max},+\infty)$.

Same atomization algorithm as before $\Rightarrow$ Initial positions of $N+1$ particles $\bar{x}_0 < \ldots < \bar{x}_N$.

[2] DF, Fagioli, Radici - submitted

The discrete model

$$\dot{x}_i(t) = -\frac{1}{N} \underbrace{v(R_i(t))}_{\text{forward density}} \sum_{j>i} K'(x_i(t) - x_j(t)) - \frac{1}{N} \underbrace{v(R_{i-1}(t))}_{\text{backward density}} \sum_{j<i} K'(x_i(t) - x_j(t))$$

Discrete density: $$R_i(t) = \frac{1}{N(x_{i+1}(t) - x_i(t))}$$

Properties:

- Discrete maximum principle $x_{i+1}(t) - x_i(t) \geq \frac{M}{\rho_{\max} N}$, i.e. $R_i(t) \leq \rho_{\max}$
- Global existence
- $x_0(t) \geq \bar{x}_0$, $x_N(t) \leq \bar{x}_N$ (confined support)

Estimates

$$\rho^N(t,x) = \sum_{i=0}^{N-1} R_i(t)\chi_{[x_i(t),x_{i+1}(t))}$$

$$TV[\rho^N(\cdot,t)] = R_0(t) + \sum_{i=0}^{N-2} |R_{i+1}(t) - R_i(t)| + R_{N-1}(t)$$

Uniform *BV* estimate

There exists a constant $C > 0$ depending only on K, v, and $\mathrm{meas}(\mathrm{supp}[\bar{\rho}])$, such that

$$TV[\rho^N(\cdot,t)] \leq TV[\bar{\rho}]e^{Ct}, \quad \text{for all } t \geq 0.$$

- The proof crucially uses the monotonicity of v and the splitting of the use of 'upwind' densities in the velocity field.
- The overcrowding prevention effect is also crucial: without it, particles would collapse into a single point mass.

Convergence of the scheme

- The previous BV-estimate allows to control space-oscillations.
- As for time-oscillations, we prove Lipschitz equi-continuity in the 1-Wasserstein distance
$$W_1(\rho^N(t,\cdot),\rho^N(s,\cdot)) \leq C|t-s|,$$
with $C>0$ independent of N.
- Rossi-Savaré 2003 (Aubin/Lions-type compactness theorem) implies strong compactness of ρ^N in $L^1([0,+\infty)\times\mathbb{R})$.

Entropy solutions

Similarly to scalar conservation laws, we define

Definition

$\rho : [0,+\infty) \times \mathbb{R} \to [0,+\infty)$ is an entropy solution to (1a) with initial condition $\bar{\rho}$ if $\rho \in L^\infty([0,+\infty); L^1 \cap L^\infty(\mathbb{R}))$ and, for all constants $c \geq 0$ and for all $\varphi \in C_c([0,+\infty) \times \mathbb{R})$ with $\varphi \geq 0$ one has

$$\int_{\mathbb{R}} |\bar{\rho}(x) - c|\varphi(0,x)dx + \int_0^{+\infty} \int_{\mathbb{R}} \Big(|\rho - c|\varphi_t \\ - \mathrm{sign}(\rho - c) \left[(f(\rho) - f(c))K' * \rho\varphi_x - f(c)K'' * \rho\varphi \right] \Big)\, dxdt \geq 0,$$

where $f(z) = zv(z)$.

Notice that entropy solutions are weak solutions.

Theorem

- There exists no more than one entropy solution to (1a) with initial condition $\bar{\rho}$
- $\rho^N \to \rho$ as $N \to +\infty$ and ρ is an entropy solution (proof: very technical).

Non uniqueness of weak solutions

Consider $v(\rho) = (1-\rho)_+$ and the initial condition

$$\bar{\rho}(x) = \chi_{[-1,-1/2]} + \chi_{[1/2,1]}.$$

Let $\rho_s(t,x) = \bar{\rho}(x)$ for all $t \geq 0$.

- ρ_s is a (stationary) weak solution to (1a)
- ρ_s is *not* an entropy solution. Proof: use test functions that concentrate around $-1/2$ and $1/2$ to violate the entropy condition. Extra assumption needed: $K'' > 0$ on the support of $\bar{\rho}$.
- We know the scheme converges to an entropy solution, therefore there are *at least two weak solutions* with this initial condition.
- Why is ρ_s not satisfying the entropy condition: the discontinuities at $\pm 1/2$ are *not admissible*.
- The scheme catches this behavior because particles at $\pm 1/2$ are forced to *move*.

Simulations

Extension to the diffusive case[3]

$$\partial_t \rho = \partial_{xx}\varphi(\rho) + \partial_x(\rho v(\rho) K' * \rho)$$

- On bounded intervals with no-slip boundary conditions
- Initial datum away from the vacuum state
- Convergence to weak solutions via BV estimate
- Diffusion term may be degenerate, φ is required to be *non-decreasing*
- Assumption on K are the same as before.

[3] Fagioli, Radici - to appear on M3AS

End of the talk

Thanks for your attention!