# A Theory of Transfers

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# Notation

$X$ and $Y$ are compact spaces. $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and weak$^*$ lower semi-continuous on $\mathcal{M}(X) \times \mathcal{M}(Y)$. $D(\mathcal{T})$ will denote its effective domain. Its "partial domains" are then

$$D_1(\mathcal{T}) = \{\mu \in \mathcal{P}(X); \exists \nu \in \mathcal{P}(Y), (\mu, \nu) \in D(\mathcal{T})\}$$

and

$$D_2(\mathcal{T}) = \{\nu \in \mathcal{P}(Y); \exists \mu \in \mathcal{P}(X), (\mu, \nu) \in D(\mathcal{T})\}.$$

- For $\mu \in \mathcal{P}(X)$, consider the partial maps $\mathcal{T}_\mu : \nu \to \mathcal{T}(\mu, \nu)$ on $\mathcal{P}(Y)$,
- For $\nu \in \mathcal{P}(Y)$), consider the partial map $\mathcal{T}_\nu : \mu \to \mathcal{T}(\mu, \nu)$ on $\mathcal{P}(X)$,
- $+\infty$ outside the probability measures.

They are clearly convex and weak$^*$-lower semi-continuous on $\mathcal{M}(Y)$ (resp., $\mathcal{M}(X)$).

# Linear Transfers

Let $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and weak* lower semi-continuous on $\mathcal{M}(X) \times \mathcal{M}(Y)$. Say that

1. $\mathcal{T}$ is a *backward linear Transfer*, if there exists a convex operator $T^- : C(Y) \to LSC(X)$ such that for each $\mu \in D_1(\mathcal{T})$, the Legendre transform of $\mathcal{T}_\mu$ on $\mathcal{M}(Y)$ satisfies:

$$\mathcal{T}_\mu^*(g) = \int_X T^- g(x)\, d\mu(x) \quad \text{for any } g \in C(Y). \tag{1}$$

2. $\mathcal{T}$ is a *forward linear transfer*, if there exists a concave operator $T^+ : C(X) \to USC(Y)$ such that for each $\nu \in D_2(\mathcal{T})$, the Legendre transform of $\mathcal{T}_\nu$ on $\mathcal{M}(X)$ satisfies:

$$\mathcal{T}_\nu^*(f) = -\int_Y T^+(-f)(y)\, d\nu(y) \quad \text{for any } f \in C(X). \tag{2}$$

We shall call $T^+$ (resp., $T^-$) the *forward (resp., backward) Kantorovich operator* associated to $\mathcal{T}$.

# Familiar formulae

So, if $\mathcal{T}$ is a forward linear transfer on $X \times Y$, then for any $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, we have

$$\mathcal{T}(\mu, \nu) = \sup \Big\{ \int_Y T^+ f(y)\, d\nu(y) - \int_X f(x)\, d\mu(x);\ f \in C(X) \Big\},$$

while if $\mathcal{T}$ is a backward linear transfer on $X \times Y$, then

$$\mathcal{T}(\mu, \nu) = \sup \Big\{ \int_Y g(y)\, d\nu(y) - \int_X T^- g(x)\, d\mu(x);\ g \in C(Y) \Big\}.$$

A *a transfer* $\mathcal{T}$ *is symmetric* if

$$\mathcal{T}(\nu, \mu) := \mathcal{T}(\nu, \mu) \text{ for all } \mu \in \mathcal{P}(X) \text{ and } \nu \in \mathcal{P}(X).$$

If $\mathcal{T}$ is a backward linear transfer with Kantorovich operator $T^-$, then $\tilde{\mathcal{T}}(\mu, \nu) := \mathcal{T}(\nu, \mu)$ is a forward linear transfer with operator $\tilde{T}^+ f = -T^-(-f)$.
This means that if $\mathcal{T}$ is symmetric, then $T^+ f = -T^-(-f)$.

# First examples of linear mass transfers

1. The identity transfer $\mathcal{I}$ on $\mathcal{P}(X) \times \mathcal{P}(X)$ is the map

$$\mathcal{I}(\mu, \nu) = \begin{cases} 0 & \text{if } \mu = \nu \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

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2. The trivial transfer: Any pair of functions $c_1 \in C(X)$, $c_2 \in C(Y)$ define trivially a linear transfer via

$$\mathcal{T}(\mu, \nu) = \int_Y c_2 \, d\nu - \int_X c_1 \, d\mu.$$

   The Kantorovich operators are then

$$T^+ f = c_2 + \inf(f - c_1) \text{ and } T^- g = c_1 + \sup(g - c_2).$$

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$$T^+ f = c_2 + \inf(f - c_1) \text{ and } T^- g = c_1 + \sup(g - c_2).$$

3. Monge-Kantorovich transfer: Any function $c \in C(X \times Y)$ determines a linear transfer. Optimal transport theory.

$$\mathcal{T}_c(\mu,\nu) := \inf\Big\{\int_{X\times Y} c(x,y))\,d\pi; \pi\in\mathcal{K}(\mu,\nu)\Big\},$$

where $\mathcal{K}(\mu,\nu)$ is the set of probability measures $\pi$ on $X\times Y$ whose marginal on $X$ (resp. on $Y$) is $\mu$ (resp., $\nu$) .
Monge-Kantorovich theory readily yields that $\mathcal{T}_c$ is both a forward and backward linear transfer. The Kantorovich operators are:

$$T_c^+f(y) = \inf_{x\in X} c(x,y) + f(x) \quad \text{and} \quad T_c^-g(x) = \sup_{y\in Y} g(y) - c(x,y),$$

for any $f\in C(X)$ (resp., $g\in C(Y)$), and

$$\begin{aligned}\mathcal{T}_c(\mu,\nu) &= \sup\Big\{\int_Y T_c^+f(y)\,d\nu(y) - \int_X f(x)\,d\mu(x);\ f\in C(X)\Big\}\\ &= \sup\Big\{\int_Y g(y)\,d\nu(y) - \int_X T_c^-g(x)\,d\mu(x);\ g\in C(Y)\Big\}.\end{aligned}$$

# Backward and forward Hamilton-Jacobi Equations

On a given compact manifold $M$, consider the cost:

$$c^L(y,x) := \inf\{\int_0^1 L(t,\gamma(t),\dot\gamma(t))\,dt; \gamma \in C^1([0,T),M); \gamma(0)=y, \gamma(T)=x\},$$

where $L : TM \to \mathbb{R} \cup \{+\infty\}$ is a given Tonelli Lagrangian.

$$\mathcal{T}_{c^L}(\mu,\nu) := \inf\left\{\int_{M\times M} c^L(y,x)\,d\pi; \pi \in \mathcal{K}(\mu,\nu)\right\}$$

is a forward linear transfer with Kantorovich operator given by $T_1^+ f(x) = V_f(1,x)$, where $V_f(t,x)$ is –at least formally– a solution for the Hamilton-Jacobi equation

$$\begin{cases} \partial_t V + H(t,x,\nabla_x V) & = & 0 \text{ on } [0,1]\times M, \\ V(0,x) & = & f(x). \end{cases}$$

Similarly, it has a backward Kantorovich potential is given by $T_1^- g(y) = W_g(0,y)$, $W_g(t,y)$ being is a solution for the backward Hamilton-Jacobi equation

$$\begin{cases} \partial_t W + H(t,x,\nabla_x W) & = & 0 \text{ on } [0,1]\times M, \\ W(1,y) & = & g(y). \end{cases}$$

# One-sided linear transfers: constrained mass transports

1. Martingale transports:

$$\mathcal{T}_M(\mu,\nu) = \begin{cases} \inf\{\int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y)\,d\pi(x,y); \pi \in MT(\mu,\nu)\} & \text{if } \mu \prec_C \nu \\ +\infty & \text{if not.} \end{cases}$$

where $MT(\mu,\nu)$ is the set of *martingale transport plans*, i.e., probabilities $\pi$ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu$, such that for $\mu$-almost $x \in \mathbb{R}^d$, the component $\pi_x$ of its disintegration $(\pi_x)_x$ with respect to $\mu$, i.e. $d\pi(x,y) = d\pi_x(y)d\mu(x)$, and $\pi_x$ has its barycenter at $x$.
It is a backward linear transfer with Kantorovich operator:

$$T^- f(x) = f_{c,x}(x) \text{ the concave envelope of } y \to f(y) + c(x,y),$$

2. Dynamic mass transports with free-end time
Ghoussoub-Young Heon Kim-Aaron Palmer (Tomorrow).

# Non cost-minimizing transfers-Stochastic transports

Given a Lagrangian $L : [0, T] \times M \times M^* \to \mathbb{R}$, define the following stochastic counterpart of the optimal transportation problem.

$$\mathcal{T}_L(\mu,\nu) := \inf\left\{\mathbb{E}\left[\int_0^1 L(t, X(t), \beta_X(t,X))\,dt\right]\Big| X(0)\sim\mu, X(1)\sim\nu, X\in\mathcal{A}\right\}$$

$\mathcal{A}$ is the set of $\mathbb{R}^d$-valued continuous semimartingales $X(\cdot)$ verifying

$$dX = \beta_X(t)dt + dW$$

for some measurable drift $\beta_X : [0, T] \times C([0,1]) \to M^*$.
This does not fit in the standard optimal mass transport theory. However,

$$\mathcal{T}_L(\mu,\nu) = \sup\left\{\int_M f(x)\,d\nu - \int_M V_f(0,x)\,d\mu;\ f\in\mathcal{C}_b^\infty\right\},$$

where $V_f$ solves the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\Delta V(t,x) + H(t,x,\nabla V) = 0, \quad V(1,x) = f(x). \qquad \text{(HJB)}$$

$\mathcal{T}_L$ is a backward linear transfer with operator $T_L^- f = V_f(0,\cdot)$.

# Weak optimal transports

Story started with Marton who defined transports of the following type:

$$\mathcal{T}_{\gamma,d}(\mu,\nu) = \inf\left\{\int_X \gamma\left(\int_Y d(x,y)d\pi_x(y)\right)\,d\mu(x); \pi \in \mathcal{K}(\mu,\nu)\right\},$$

where $\gamma$ is convex on $\mathbb{R}^+$ and $d : X \times Y \to \mathbb{R}$ is lower semi-continuous.
Marton's weak transfer correspond to $\gamma(t) = t^2$ and $d(x,y) = |x-y|$.
This is a backward linear transfer with Kantorovich potential

$$T^- f(x) = \sup\left\{\int_Y f(y)d\sigma(y) - \gamma\left(\int_Y d(x,y)\,d\sigma(y)\right);\ \sigma \in \mathcal{P}(Y)\right\}.$$

Gozlan et al. defined Weak Transport associated to $c : X \times \mathcal{P}(X) \to \mathbb{R}$ as

$$\mathcal{T}(\mu,\nu) = \inf_\pi\{\int_X c(x,\pi_x)\,d\mu(x); \pi \in \mathcal{K}(\mu,\nu)\}.$$

# A representation of linear transfers as generalized optimal mass transports

## Proposition

*Let $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ be such that $\{\delta_x ; x \in X\} \subset D_1(\mathcal{T})$. Then, $\mathcal{T}$ is a backward linear transfer if and only if there exists a lower semi-continuous function $c : X \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ with $\sigma \to c(x, \sigma)$ convex on $\mathcal{P}(Y)$ for each $x \in X$ such that for every $\mu \in \mathcal{P}(X)$, and $\nu \in \mathcal{P}(Y)$, we have*

$$\mathcal{T}(\mu, \nu) = \inf_{\pi} \int_X c(x, \pi_x)\, d\mu(x); \pi \in \mathcal{K}(\mu, \nu)\}.$$

*The corresponding backward Kantorovich operator is given for every $g \in C(Y)$ by*

$$T^- g(x) = \sup\{\int_Y g(y)\, d\sigma(y) - \mathcal{T}(x, \sigma); \sigma \in \mathcal{P}(Y)\}.$$

# Operations on linear mass transfers

The class of backward linear transfers on $X \times Y$ is a convex cone of weak$^*$-lower semi-continuous convex functions on $\mathcal{P}(X) \times \mathcal{P}(Y)$.

1. (Inf-convolution) If $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) is a backward linear transfer on $X_1 \times X_2$ (resp., on $X_2 \times X_3$) with Kantorovich operator $T_1^-$ (resp., $T_2^-$), then

$$\mathcal{T}_1 \star \mathcal{T}_2(\mu, \nu) := \inf\{\mathcal{T}_1(\mu, \sigma) + \mathcal{T}_2(\sigma, \nu);\, \sigma \in \mathcal{P}(X_2)\}.$$

   is also a backward linear transfer on $X_1 \times X_3$ with Kantorovich operator equal to $T_1^- \circ T_2^-$.

2. (Tensorization) If $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) is a backward linear transfer on $X_1 \times Y_1$ (resp., $X_2 \times Y_2$) with $X_1 \subset D(\mathcal{T}_1)$ and $X_2 \subset D(\mathcal{T}_2)$, then

$$\mathcal{T}_1 \otimes \mathcal{T}_2(\mu, \nu) = \inf\left\{\int_{X_1 \times X_2} \left(\mathcal{T}_1(x_1, \pi_{x_1,x_2}) + \mathcal{T}_2(x_2, \pi_{x_1,x_2})\right) d\mu(x_1, x_2); \pi \in \mathcal{K}(\mu, \nu)\right\}.$$

   is a backward linear transfer on $(X_1 \times X_2) \times (Y_1 \times Y_2)$, with Kantorovich operator

$$T^- g(x_1, x_2) = \sup\{\int_{Y_1 \times Y_2} f(y_1, y_2) d\sigma(y_1, y_2) - \mathcal{T}_1(x_1, \sigma_1) - \mathcal{T}_2(x_2, \sigma_2);\, \sigma \in \mathcal{K}(\sigma_1, \sigma_2)\}.$$

# Convex Transfers

$\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ is said to be a *backward convex transfer* (resp., *forward convex transfer*), if there exists a family of backward linear transfers (resp., forward linear transfers) $(\mathcal{T}_i)_{i \in I}$ such that

$$\mathcal{T}(\mu, \nu) = \sup_{i \in I} \mathcal{T}_i(\mu, \nu) \quad \text{for all } \mu \in \mathcal{P}(X),\ \nu \in \mathcal{P}(Y).$$

1. $\mathcal{T}$ is a *backward convex transfer*, if there exists a family of convex operators $(T_i^-)_{i \in I}$ from $C(Y) \to LSC(X)$ such that for each $\mu \in D_1(\mathcal{T})$, the Legendre transform of $\mathcal{T}_\mu$ on $\mathcal{M}(Y)$ satisfies:

$$\mathcal{T}_\mu^*(g) = \inf_{i \in I} \int_X T_i^- g(x)\, d\mu(x) \quad \text{for any } g \in C(Y).$$

2. $\mathcal{T}$ is a *forward convex transfer*, if there exists a family of concave operators $(T_i^+)_s$ from $C(X) \to USC(Y)$ such that for each $\nu \in D_2(\mathcal{T})$, the Legendre transform of $\mathcal{T}_\nu$ on $\mathcal{M}(X)$ satisfies:

$$\mathcal{T}_\nu^*(f) = -\sup_{i \in I} \int_Y T_i^+(-f)(y)\, d\nu(y) \quad \text{for any } f \in C(X).$$

# Examples of convex transfers

1. If $\mathcal{T}$ is a linear backward (resp., forward) transfer and $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is convex increasing, then $\alpha(\mathcal{T})$ is a backward (resp., forward) convex transfer.
2. In particular, for any $p \geq 1$, $\mathcal{T}^p$ is a convex transfer.
3. If $\alpha$ is a strictly convex and superlinear, then
$$\mathcal{T}(\mu,\nu) = \int_X \alpha(\frac{d\nu}{d\mu})\, d\mu, \quad \text{if } \mu << \nu \text{ and } +\infty \text{ otherwise.}$$
is a backward convex transfer.
4. The *Donsker-Varadhan entropy*, which is defined as
$$\mathcal{I}(\mu,\nu) := \begin{cases} \mathcal{E}(\sqrt{f},\sqrt{f}), & \text{if } \mu = f\nu, \sqrt{f} \in \mathbb{D}(\mathcal{E}) \\ +\infty, & \text{otherwise,} \end{cases}$$
where $\mathcal{E}$ is a Dirichlet form with domain $\mathbb{D}(\mathcal{E})$ on $L^2(\nu)$, is a backward convex transfer.

# Entropic Transfers: An important class of convex transfers

Let $\alpha$ (resp., $\beta$) be a convex increasing (resp., concave increasing) real function on $\mathbb{R}$, and let $\mathcal{E} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$. Say that

- $\mathcal{E}$ is a *$\beta$-backward transfer*, if there exists a convex operator $E^- : C(Y) \to LSC(X)$ such that for each $\mu \in D_1(\mathcal{T})$, the Legendre transform of $\mathcal{E}_\mu$ on $\mathcal{M}(Y)$ is:

$$\mathcal{E}_\mu^*(g) = \beta\left(\int_X E^- g(x)\, d\mu(x)\right) \quad \text{for any } g \in C(Y).$$

- $\mathcal{E}$ is a *$\alpha$-forward transfer*, if there exists a concave operator $E^+ : C(X) \to USC(Y)$ such that for each $\nu \in D_2(\mathcal{T})$,

$$\mathcal{E}_\nu^*(f) = -\alpha\left(\int_Y E^+(-f)(y)\, d\nu(y)\right) \quad \text{for any } f \in C(X).$$

If $\mathcal{T}$ is a backward linear transfer with Kantorovich operator $T^-$, then $\mathcal{E} \star \mathcal{T}$ is a a backward $\beta$-transfer with Kantorovich operator $E^- \circ T^-$.

$$\mathcal{E}\star\mathcal{T}(\mu,\nu) = \sup\left\{ \int_Z g(y)\, d\nu(y) - \beta\left(\int_X E^- \circ T^- g(x)\right) d\mu(x)); \; g \in C(X_3)\right\}.$$

# Logarithmic Transfers

If $\mathcal{E}$ is an $\alpha$-forward transfer on $X \times Y$, then for $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$,

$$\mathcal{E}(\mu,\nu) = \sup\left\{\alpha\left(\int_Y E^+ f(y)\, d\nu(y)\right) - \int_X f(x)\, d\mu(x);\ f \in C(X)\right\},$$

while if $\mathcal{E}$ is a $\beta$-backward transfer, then

$$\mathcal{E}(\mu,\nu) = \sup\left\{\int_Y g(y)\, d\nu(y) - \beta\left(\int_X E^- g(x)\, d\mu(x)\right);\ g \in C(Y)\right\}.$$

A typical example is of course the logarithmic entropy,

$$\mathcal{H}(\mu,\nu) = \int_X \log(\frac{d\nu}{d\mu})\, d\nu, \quad \text{if } \nu << \mu \text{ and } +\infty \text{ otherwise}$$

$$\mathcal{H}(\mu,\nu) = \sup\{\int_X f\, d\nu - \log(\int_X e^f\, d\mu);\ f \in C_b(X)\},$$

making it a log-backward transfer.

# Transfer Inequalities

Standard Transport-Entropy inequalities are normally of the form

$$\mathcal{T}(\sigma,\mu) \leq \lambda_1 \mathcal{E}_1(\mu,\sigma) \quad \text{for all } \sigma \in \mathcal{P}(X),$$

$$\mathcal{T}(\mu,\sigma) \leq \lambda_2 \mathcal{E}_2(\mu,\sigma) \quad \text{for all } \sigma \in \mathcal{P}(X),$$

$$\mathcal{T}(\sigma_1,\sigma_2) \leq \lambda_1 \mathcal{E}_1(\sigma_1,\mu) + \lambda_2 \mathcal{E}_2(\sigma_2,\mu) \quad \text{for all } \sigma_1, \sigma_2 \in \mathcal{P}(X),$$

where $\mu$ is a fixed measure, and $\lambda_1$, $\lambda_2$ are two positive reals.
In our terminology, These amount to find $\mu$, $\lambda_1$, and $\lambda_2$ such that

$$(\lambda_1 \mathcal{E}_1) \star (-\mathcal{T})\,(\mu,\mu) \geq 0,$$

$$\lambda_2 \mathcal{E}_2) \star (-\tilde{\mathcal{T}})\,(\mu,\mu) \geq 0,$$

$$(\lambda_1 \tilde{\mathcal{E}}_1) \star (-\mathcal{T}) \star (\lambda_2 \mathcal{E}_2)\,(\mu,\mu) \geq 0,$$

where $\tilde{\mathcal{T}}(\mu,\nu) = \mathcal{T}(\nu,\mu)$. Note for example that

$$\tilde{\mathcal{E}}_1 \star (-\mathcal{T}) \star \mathcal{E}_2\,(\mu,\nu) = \inf\{\mathcal{E}_1(\sigma_1,\mu) - \mathcal{T}_2(\sigma_1,\sigma_2) + \mathcal{E}_2(\sigma_2,\nu);\ \sigma_1,\sigma_2 \in \mathcal{P}(Z)\}.$$

One then writes duality formulas for the transfers

$$\mathcal{E}_1 \star (-\mathcal{T}), \quad \mathcal{E}_2 \star (-\tilde{\mathcal{T}}) \quad \text{and} \quad \tilde{\mathcal{E}}_1 \star (-\mathcal{T}) \star \mathcal{E}_2$$

where $\mathcal{T}$ is any convex transfer, while $\mathcal{E}_1$, $\mathcal{E}_2$ are entropic transfers.

# A sample: Extension of Maurey's inequality

- Consider $\mathcal{E}_1$ (resp., $\mathcal{E}_2$) a forward $\alpha_1$-transfer on $Z_1 \times X_1$ (resp., $\alpha_2$-transfer on $Z_2 \times X_2$) with Kantorovich operator $E_1^+$ (resp., $E_2^+$).
- Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) be forward linear transfers on $Y_1 \times Z_1$ (resp., $Y_2 \times Z_2$) with Kantorovich operator $T_1^+$ (resp., $T_2^+$).
- Let $\mathcal{F}$ be a backward convex transfer on $Y_1 \times Y_2$ with Kantorovich operators $(F_i^-)_i$.

Then, for $\mu \in \mathcal{P}(X_1)$ and $\nu \in \mathcal{P}(X_2)$ given, TFAE:

1. For all $\sigma_1 \in \mathcal{P}(X_1), \sigma_2 \in \mathcal{P}(X_2)$, we have

$$\mathcal{F}(\sigma_1, \sigma_2) \leq \lambda_1 \mathcal{T}_1 \star \mathcal{E}_1(\sigma_1, \mu) + \lambda_2 \mathcal{T}_2 \star \mathcal{E}_2(\sigma_2, \nu).$$

2. For all $g \in C(Y_2)$ and all $i \in I$, we have

$$\lambda_1\alpha_1\Big(\int_{X_1} E_1^+ \circ T_1^+ \circ (-\frac{1}{\lambda_1} F_i^- g)\, d\mu\Big) + \lambda_2\alpha_2\Big(\int_{X_2} E_2^+ \circ T_2^+ (\frac{1}{\lambda_2} g)\, d\nu\Big) \geq 0.$$

# Weak KAM theory on Wasserstein space

Let $X$ be a compact metric space, and let $\mathcal{T}$ be a backward linear transfer on $X \times X$ with Kantorovich operator $T$. For $n \in \mathbb{N}$, Let $\mathcal{T}_n = \mathcal{T} \star \mathcal{T} \star .... \star \mathcal{T}$ $n$-times. Then

1. $\mathcal{T}_n(\mu, \nu) = \sup\left\{\int_X g(y)\, d\nu - \int_X T^n g(x)\, d\mu;\ g \in C(X)\right\}$.

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1. $\mathcal{T}_n(\mu, \nu) = \sup \left\{ \int_X g(y)\, d\nu - \int_X T^n g(x)\, d\mu;\ g \in C(X) \right\}$.
2. There exists a constant $C > 0$ and a number $\ell \in \mathbb{R}$ such that
$$|\mathcal{T}_n(\mu, \nu) - \ell n| \leq C \qquad \text{for all } \mu, \nu \in \mathcal{P}(X) \text{ and } n \in \mathbb{N}.$$
3. Weak KAM solutions: Assume $\ell = 0$, then there exists $T_\infty : C(X) \to C(X)$ such that $T T_\infty f = T_\infty f$. Moreover, $T_\infty T_\infty f = T_\infty f$.

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3. Weak KAM solutions: Assume $\ell = 0$, then there exists $T_\infty : C(X) \to C(X)$ such that $TT_\infty f = T_\infty f$. Moreover, $T_\infty T_\infty f = T_\infty f$.
4. Peierls Barrier: $\mathcal{T}_\infty(\mu, \nu) := \sup\limits_{f \in C(X)} \left\{ \int_X f d\nu - \int_X T_\infty f d\mu \right\}$ is a backward linear transfer.

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$$|\mathcal{T}_n(\mu,\nu) - \ell n| \le C \qquad \text{for all } \mu,\nu \in \mathcal{P}(X) \text{ and } n \in \mathbb{N}.$$
3. Weak KAM solutions: Assume $\ell = 0$, then there exists $T_\infty : C(X) \to C(X)$ such that $TT_\infty f = T_\infty f$. Moreover, $T_\infty T_\infty f = T_\infty f$.
4. Peierls Barrier: $\mathcal{T}_\infty(\mu,\nu) := \sup\limits_{f \in C(X)} \left\{\int_X f d\nu - \int_X T_\infty f d\mu\right\}$ is a backward linear transfer.
5. Mather measure: $\inf\limits_{\mu \in \mathcal{P}(X)} \mathcal{T}(\mu,\mu) = 0$ and the infimum is achieved by a measure $\bar{\mu}$ in the projected Aubry set
$$\mathcal{A} := \{\mu \in \mathcal{P}(X)\ :\ \mathcal{T}_\infty(\mu,\mu) = 0\}$$
such that $(\bar{\mu},\bar{\mu})$ belongs to the Aubry set
$$\mathcal{D} := \{(\mu,\nu) \in \mathcal{P}(X) \times \mathcal{P}(X)\ :\ \mathcal{T}(\mu,\nu) + \mathcal{T}_\infty(\nu,\mu) = 0\} \subset \mathcal{A} \times \mathcal{A}.$$

Multi-transfers are even more fascinating!

THANK YOU