Lagrangian fibrations by Jacobians and Prym varieties

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Higgs bundles

Let Σ be a Riemann surface of genus $g \geqslant 2$.

Defn: A $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle is a pair (E, Φ) of a (degree δ, rank n) holomorphic bundle E and a Higgs field

$$\Phi \in \mathrm{H}^0(\Sigma, K \otimes \mathrm{End}\, E).$$

It is stable if for all Φ-invariant subbundles $F \subset E$

$$\frac{\deg F}{\mathrm{rank} F} < \frac{\deg E}{\mathrm{rank} E}.$$

The moduli space $\mathcal{M}_{\mathrm{GL}}$ of stable Higgs bundles admits a holomorphic symplectic structure σ.

Rmk: $T^*\mathcal{B}un_{\mathrm{GL}} \subset \mathcal{M}_{\mathrm{GL}}$ as

$$T^*_E \mathcal{B}un_{\mathrm{GL}} = \mathrm{H}^1(\Sigma, \mathrm{End}\, E)^* \cong \mathrm{H}^0(\Sigma, K \otimes \mathrm{End}\, E).$$

The Hitchin map

We can map a Higgs bundle to the eigenvalues of the $\mathrm{End}E$-valued one-form Φ

$$\begin{aligned} h : \mathcal{M}_{\mathrm{GL}} &\longrightarrow A_{\mathrm{GL}} := \bigoplus_{i=1}^{n} \mathrm{H}^0(\Sigma, K^i) \\ (E, \Phi) &\longmapsto (\mathrm{tr}\Phi, \mathrm{tr}(\Phi^2), \ldots, \mathrm{tr}(\Phi^n)) \end{aligned}$$

The latter determine a spectral curve

$$\begin{array}{ccccc} C & \subset & \mathrm{Tot}K & = & T^*\Sigma \\ & {\scriptstyle n:1} \searrow & \downarrow & & \\ & & \Sigma & & \end{array}$$

The eigenspaces determine a line bundle L over C.

Defn: (C, L) is called the spectral data of (E, Φ).

The Hitchin system

Thm (Hitchin): $h : \mathcal{M}_{\mathrm{GL}} \longrightarrow A_{\mathrm{GL}}$ is an integrable system, i.e., $\mathrm{dim}\mathcal{M}_{\mathrm{GL}} = 2\mathrm{dim}A_{\mathrm{GL}}$ and

$$[h_i, h_j] := \sigma^{-1}(dh_i, dh_j) = 0$$

where σ^{-1} is the inverse of the symplectic structure. (Equivalently, the fibres are Lagrangian wrt σ.)

The fibres are complex tori, Jacobians $\mathrm{Jac}^d C$ of spectral curves.

Rmk: There are also singular fibres, the most singular being the nilpotent cone

$$h^{-1}(n\Sigma) := \{(E, \Phi) \mid \Phi \text{ is nilpotent}\}.$$

Special Kähler geometry

Let $M \to B$ be a Lagrangian fibration.

Thm (Freed, Hitchin): B^{reg} admits a special Kähler metric

$$\omega = \frac{-i}{2}\mathrm{Im}(\tau_{ij})dz^i \wedge d\bar{z}^j$$

where τ_{ij} are the periods of the fibres.

On A_{GL}, $z^i = \int_{a_i} \theta$ where $a_1 \dots a_g, b_1 \dots b_g$ is a symplectic basis of 1-cycles on the spectral curve, θ is the canonical 1-form on $T^*\Sigma$.

Thm (Donagi-Markman): There exists a symmetric cubic form on the base, $c \in \mathrm{H}^0(B, \mathrm{Sym}^3 T_B)$, given by $c_{ijk} = \frac{\partial \tau_{jk}}{\partial z^i}$.

Relation to topological recursion

Thm (Baraglia-Huang): For the $\mathrm{GL}(n, \mathbb{C})$-Hitchin system

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-2}} \tau_{i_{m-1} i_m}(b) = -\left(\frac{i}{2\pi}\right)^{m-1} \int_{p_1 \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_m^{(0)}(p_1, \ldots, p_m)$$

where $b \in B^{\mathrm{reg}}$ and $W_m^{(0)}$ are the $g = 0$ Eynard-Orantin invariants of the spectral curve C_b.

In particular, the special Kähler metric and the Donagi-Markman cubic on B can be computed from $W_m^{(0)}$.

Lagrangian fibrations

Let X be a compact holomorphic symplectic manifold of dimension $2n$, with σ a non-degenerate two-form: $\sigma^{\wedge n}$ trivializes $\Omega^{2n} = K$.

Assume X is irreducible, i.e., σ generates $\mathrm{H}^0(X, \Omega^2)$.

Thm (Matsushita, Hwang): If $X \to B$ is a proper fibration then

1. $\dim B = n = \dim F$,
2. F is Lagrangian, generic fibre is a complex torus,
3. B is isomorphic to $\mathbb{P}^n$ if it is smooth.

Rmk: Lagrangian means $TF \subset TX$ is maximal isotropic wrt σ. Integrable means $T^*B \subset T^*X$ is maximal isotropic wrt σ^{-1}.

The Beauville-Mukai system

Let C be a genus g curve in a K3 surface S. Then $|C| \cong \mathbb{P}^g$. Let $\mathcal{C}/\mathbb{P}^g$ be the family of curves linearly equivalent to C. Then

$$X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^g) \longrightarrow \mathbb{P}^g$$

is a Lagrangian fibration.

Rmk: $$0 \longrightarrow TX_b \longrightarrow TX|_{X_b} \longrightarrow \pi^* T_b\mathbb{P}^g \longrightarrow 0$$

The normal bundle of C in S is isomorphic to $T^*C = K_C$, which implies that $TX_b = \mathrm{H}^0(C, K_C)^\vee$ is dual to $T_b\mathbb{P}^g = \mathrm{H}^0(C, K_C)$.

X can be identified with the moduli space $M(0, [C], 1 - g + d)$ of stable sheaves on S, which is holomorphic symplectic.

Rmk: If $[C] \in \mathrm{NS}(S)$ is primitive then any d is possible, whereas if $C \in |n\Sigma|$ then only some choices of d yield compact moduli spaces.

A degeneration

If $\Sigma \subset S$ is ample then S degenerates to $\overline{T^*\Sigma}$: embed

$$S \hookrightarrow \mathbb{P}(\mathrm{H}^0(S, \Sigma)^*) = \mathbb{P}^N,$$

take the cone over S in $\mathbb{P}^{N+1}$, then intersect with the pencil of hyperplanes containing Σ.

- the generic intersection is isomorphic to S
- the hyperplane through the apex of the cone gives $\overline{T^*\Sigma}$

Thm (Donagi-Ein-Lazarsfeld): This degeneration induces a degeneration of the Beauville-Mukai system built from $|n\Sigma|$ to a compactification $\overline{\mathcal{M}}_{\mathrm{GL}}$ of the $\mathrm{GL}(n, \mathbb{C})$-Hitchin system on Σ.

Curves in $|n\Sigma|$ in S become spectral curves in $\overline{T^*\Sigma}$.

Relation to topological recursion?

Qu: Can we compute the Donagi-Markman cubic and special Kähler metric on $(\mathbb{P}^g)^{\mathrm{reg}}$ from $W_m^{(0)}$ of the spectral curves C?

In both $T^*\Sigma$ and S, the normal bundle to C is $T^*C = K_C$, and

$$0 \longrightarrow TC \longrightarrow TS|_C \longrightarrow K_C \longrightarrow 0$$

gives an extension class in

$$\mathrm{Ext}^1(K_C, TC) \cong \mathrm{H}^0(C, K_C^{\otimes 3})^\vee \longrightarrow \mathrm{H}^0(C, K_C)^{\otimes 3 \vee} \cong (T_b \mathbb{P}^g)^{\otimes 3 \vee}.$$

This is the Donagi-Markman cubic at $b \in \mathbb{P}^g$.

But to higher order, the neighborhoods of C in $T^*\Sigma$ and S differ.

Qu: Is this encoded in $\theta|_C$, and therefore, in the $W_m^{(0)}$?

SL-Hitchin systems

For $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles (E,Φ), $\det E \cong \mathcal{O}$ and $\mathrm{tr}\Phi = 0$, so

$$h : \mathcal{M}_{\mathrm{SL}} \longrightarrow A_{\mathrm{SL}} := \bigoplus_{i=2}^{n} \mathrm{H}^0(\Sigma, K^i).$$

Recall the spectral curves are $n:1$ covers $C \to \Sigma$. This induces

$$\mathrm{Nm} : \mathrm{Jac}^d C \longrightarrow \mathrm{Jac}^d \Sigma$$

and the fibres of h are the Prym varieties $\mathrm{Nm}^{-1}(0)$.

Rmk: The cover $C \to \Sigma$ has branch points, so $\mathrm{Prym}(C/\Sigma)$ is not principally polarized.

PGL-Hitchin systems

For $\mathrm{PGL}(n, \mathbb{C})$-Higgs bundles we also have

$$h : \mathcal{M}_{\mathrm{PGL}} \longrightarrow A_{\mathrm{PGL}} := \bigoplus_{i=2}^{n} \mathrm{H}^0(\Sigma, K^i).$$

Now $C \to \Sigma$ induces $\mathrm{Jac}^0\Sigma \to \mathrm{Jac}^0 C$ by pullback. The fibres of h are the quotients of $\mathrm{Jac}^0 C$ by the action of $\mathrm{Jac}^0\Sigma$.

Thus $\mathcal{M}_{\mathrm{PGL}}/A_{\mathrm{PGL}}$ is the dual fibration of $\mathcal{M}_{\mathrm{SL}}/A_{\mathrm{SL}}$:

$$\mathrm{SL}: \qquad 0 \longrightarrow \mathrm{Prym}(C/\Sigma) \longrightarrow \mathrm{Jac}^d C \xrightarrow{\mathrm{Nm}} \mathrm{Jac}^d\Sigma \longrightarrow 0$$

$$\mathrm{PGL}: \qquad 0 \longrightarrow \mathrm{Jac}^0\Sigma \longrightarrow \mathrm{Jac}^0 C \longrightarrow \mathrm{Prym}(C/\Sigma)^* \longrightarrow 0$$

Thm (Hausel-Thaddeus): The stringy Hodge numbers of $\mathcal{M}_{\mathrm{PGL}}$ equal the Hodge numbers of $\mathcal{M}_{\mathrm{SL}}$.

Sp-Hitchin systems

For $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundles the spectral curves $C \subset T^*\Sigma$ are invariant under fibre multiplication by -1. Thus

$$h : \mathcal{M}_{\mathrm{Sp}} \longrightarrow A_{\mathrm{Sp}} := \bigoplus_{i=1}^{n} \mathrm{H}^0(\Sigma, K^{2i}).$$

Quotienting by the involution $\eta \mapsto -\eta$ in the fibre of $T^*\Sigma$ gives

$$\begin{array}{ccc} C & \subset & \mathrm{Tot}K = T^*\Sigma \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & \mathrm{Tot}K^2. \end{array}$$

Thus the spectral curves are (branched) double covers $C \to D$.

Generalized Prym varieties

Def: The map $\pi : C \to D$ induces $\mathrm{Nm} : \mathrm{Jac}^d C \to \mathrm{Jac}^d D$ and we define the Prym variety $\mathrm{Prym}(C/D) := \mathrm{Nm}^{-1}(0)$.

Equivalently, let $\tau : C \to C$ be the covering involution. For $d = 0$

$$\mathrm{Prym}(C/D) := \mathrm{Fix}(-\tau^*)^0 \subset \mathrm{Jac}^0 C.$$

Rmk: Compare to $\mathrm{Fix}(\tau^*)^0 \cong \pi^* \mathrm{Jac}^0 D \subset \mathrm{Jac}^0 C$.

$\mathrm{Prym}(C/D)$ has dimension genus$C$ − genus$D$ and polarization of type $(1, \ldots, 1, 2, \ldots, 2)$ with genus$D$ $2s$.

Prop: For $\mathrm{Sp}(2n, \mathbb{C})$, the fibres of h are $\mathrm{Prym}(C/D)$.

SO-Hitchin systems

For $\mathrm{SO}(2n+1,\mathbb{C})$-Higgs bundles the spectral curves consist of (-1)-invariant $C \subset T^*\Sigma$ as in the $\mathrm{Sp}(2n,\mathbb{C})$-system union with the zero section.

$$h : \mathcal{M}_{\mathrm{SO}(2n+1,\mathbb{C})} \longrightarrow A_{\mathrm{SO}(2n+1,\mathbb{C})} := \bigoplus_{i=1}^{n} \mathrm{H}^0(\Sigma, K^{2i})$$

Discarding the zero section we get $C \xrightarrow{2:1} D$ as before.

Fibres of h are finite covers of $\mathrm{Prym}(C/D)$, in fact $\mathrm{Prym}(C/D)^\vee$.

Thus $\mathcal{M}_{\mathrm{SO}(2n+1,\mathbb{C})}/A_{\mathrm{SO}(2n+1,\mathbb{C})}$ is the dual fibration of $\mathcal{M}_{\mathrm{Sp}}/A_{\mathrm{Sp}}$.

Rmk: In general $A_{\mathrm{G}} = A_{{}^{\mathrm{L}}\mathrm{G}}$ and the dual of $\mathcal{M}_{\mathrm{G}}/A_{\mathrm{G}}$ is $\mathcal{M}_{{}^{\mathrm{L}}\mathrm{G}}/A_{{}^{\mathrm{L}}\mathrm{G}}$ where ${}^{\mathrm{L}}\mathrm{G}$ is the Langlands dual group of G.

Markushevich-Tikhomirov system

Let $S \to T$ be a K3 double cover of a degree two del Pezzo.

We get a $\mathbb{P}^2$-family of genus three curves covering elliptic curves

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

The family $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ over $\mathbb{P}^2$ is a Lagrangian fibration.

Total space is a holomorphic symplectic orbifold of dimension four.

Rmk: The fibres have polarization type $(1, 2)$.

The dual fibration

The double cover $S \to T$ is constructed from two quartics Δ and Δ' in $\mathbb{P}^2$, which are tangent to each other at eight points.

- $f : T \to \mathbb{P}^2$ is a double cover branched over Δ
- $S \to T$ is branched over one component of $f^{-1}(\Delta')$

Interchanging Δ and Δ' gives $S' \to T'$.

Thm (Menet): $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ over $\mathbb{P}^2$ is dual to $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$.

Pantazis's bigonal construction

Given a tower of branched covers $C \xrightarrow{2:1} D \xrightarrow{2:1} \mathbb{P}^1$ we can construct another tower $C' \xrightarrow{2:1} D' \xrightarrow{2:1} \mathbb{P}^1$ as follows.

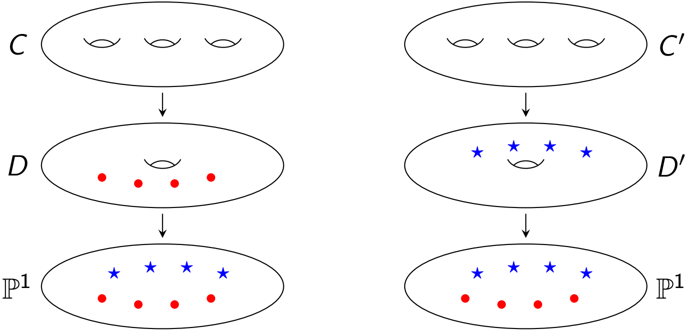
- suppose $d_1, d_2 \in D$ sit above $p \in \mathbb{P}^1$
- suppose $c_{11}, c_{12} \in C$ sit above d_1 and $c_{21}, c_{22} \in C$ sit above d_2
- above p, C' consists of pairs of lifts $\{c_{11}, c_{21}\}$, $\{c_{11}, c_{22}\}$, $\{c_{12}, c_{21}\}$, $\{c_{12}, c_{22}\}$
- an involution $\{c_{11}, c_{21}\} \leftrightarrow \{c_{12}, c_{22}\}$, $\{c_{11}, c_{22}\} \leftrightarrow \{c_{12}, c_{21}\}$
- quotienting C' by the involution gives D'

Thm (Pantazis): $\mathrm{Prym}(C'/D')$ is dual to $\mathrm{Prym}(C/D)$.

Rmk: Let $D/\mathbb{P}^1$ be branched over $p_1, \ldots, p_{2s}$ and C/D be branched over points whose images in $\mathbb{P}^1$ are $q_1, \ldots, q_{2t}$. Then for $C' \to D' \to \mathbb{P}^1$ the roles of p_i and q_j are reversed.

Dual Prym varieties

The interchange of branch points:



Menet applies Pantazis's bigonal construction to families of curves in $S/T/\mathbb{P}^2$ and $S'/T'/\mathbb{P}^2$, to conclude that their Markushevich -Tikhomirov systems are dual.

Matteini system

Let $S \to T$ be a K3 double cover of a cubic del Pezzo (degree 3).

We get a $\mathbb{P}^3$-family of genus four curves covering elliptic curves

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

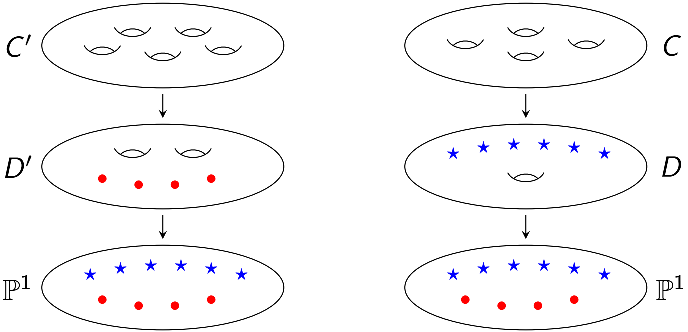
The family $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ over $\mathbb{P}^3$ is a Lagrangian fibration.

Total space is a holomorphic symplectic orbifold of dimension six.

Rmk: The fibres have polarization type $(1,1,2)$.

Pantazis's bigonal construction again

Fibrewise the dual Prym varieties arise from:



The LHS curves $C' \to D' \to \mathbb{P}^1$ come from a new system.

Conjectural dual of Matteini's system

[joint work with Chen Shen]

Let $S' \to T' \to Q$ be a K3 double cover of a degree one del Pezzo.

Get a $\mathbb{P}^3$-family of genus five curves covering genus two curves and $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ is a holomorphic symplectic orbifold of dimension six.

Rmk: Lagrangian fibration with fibres of polarization type $(1, 2, 2)$.

Conj: Our system is dual to Matteini's.

A parameter count suggests that one may need to specialize Matteini's system.

Known examples of compact Prym fibrations

Thm (Nikulin): There are 75 anti-symplectic involutions on K3s. The quotient $T := S/\tau$ is an Enriques or a rational surface.

Lagrangian fibrations by Pryms:

- **Markushevich-Tikhomirov:** K3 cover of degree 2 del Pezzo,
- **Arbarello-Saccà-Ferretti:** K3 cover of Enriques surfaces,
- **Matteini:** K3 covers of other del Pezzos and Hirzebruchs,
- **Debarre:** linear systems of curves in abelian surfaces.
- **Matteini:** abelian double covers of bielliptic surfaces.

Degenerations

[*joint work with Chen Shen*]

To connect the Beauville-Mukai system to the $\mathrm{GL}(n, \mathbb{C})$-Hitchin system we started with a degeneration:

$$S \rightsquigarrow \overline{T^*\Sigma}$$

For Prym fibrations, we start with a degeneration of double covers:

$$\begin{array}{ccc} S & \rightsquigarrow & \overline{\mathrm{Tot}K} = \overline{T^*\Sigma} \\ \downarrow & & \downarrow \\ T & \rightsquigarrow & \overline{\mathrm{Tot}K^2} \end{array}$$

Rmk: The branch locus of S/T becomes the branch locus of $\mathrm{Tot}K/\mathrm{Tot}K^2$, which is just the zero section $\cong \Sigma$.

Degenerations

Induces degenerations of *some* Prym fibrations coming from K3 covers of del Pezzos to compactifications of Sp-Hitchin systems.

Rmk: For Hitchin systems the spectral curves lie in $|n\Sigma|$, and Σ is the branch locus of $\mathrm{Tot}K/\mathrm{Tot}K^2$.

Thus a compact system that generates to it must have $C \in |n\Delta|$.

(Not true for the Markushevish-Tikhomirov and Matteini systems.)

Summary of Lagrangian fibrations

Fibres	**Non-compact**	**Compact**
Jacobians	$\mathrm{GL}(n, \mathbb{C})$-Hitchin	Beauville-Mukai
Prym varieties	$\mathrm{Sp}(2n, \mathbb{C})$-Hitchin	Markushevich-Tikhomirov
	$\mathrm{SO}(2n+1, \mathbb{C})$-Hitchin	Matteini
		⋮

Thank you!