Thin Trees

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Stanford University

based on joint work with

Shayan Oveis Gharan

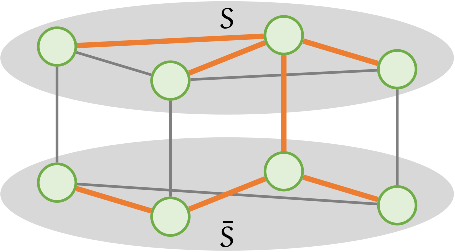
Thin Tree Recap

Thinness

T is α-thin w.r.t. G iff

$$|T(S, \bar{S})| \leqslant \alpha \cdot |G(S, \bar{S})|,$$

for every subset of vertices S.



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or in other words for every $x \in \mathbb{R}^n$,

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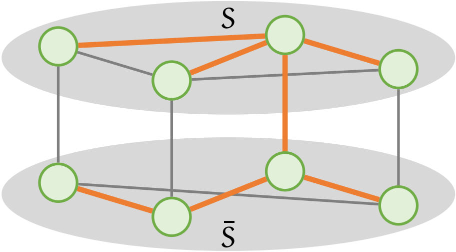

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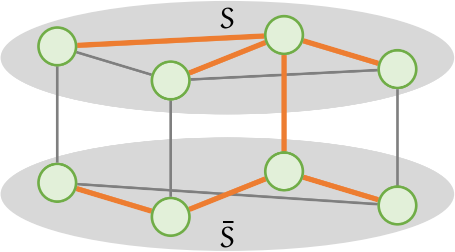

α-spectrally thin

$\implies$ α-thin

Thin Tree Conjecture

Strong Form of [Goddyn]

Every k-edge connected graph has $O(1/k)$-thin spanning tree.

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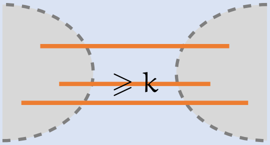
- This implies $O(1)$ upper bound for integrality gap of LP relaxation for ATSP.
- Existence of $f(n)/k$-thin trees implies $O(f(n))$ upper bound for integrality gap of LP relaxation for ATSP.
- $O(1)$ integrality gap already proved [Svensson-Tarnawski-Végh'17], but thin tree remains open.

Spectral Thinness

Edge Connectivity

$$|G(S, \bar{S})| \geqslant k$$



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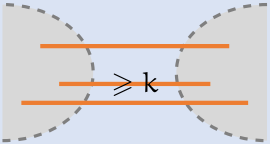

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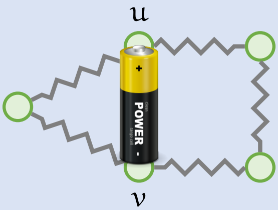
Electrical Connectivity

$$\mathrm{Reff}(u, v) \leqslant \frac{1}{k}$$



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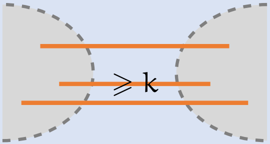

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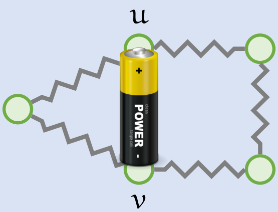

[Harvey-Olver'14, Marcus-Spielman-Srivastava'14]

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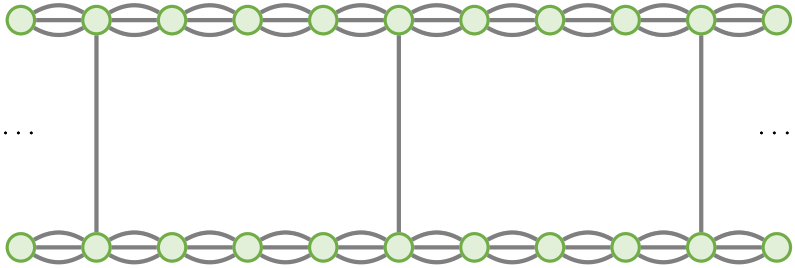
Obstacles

- Problem: Edge connectivity does not imply electrical connectivity.



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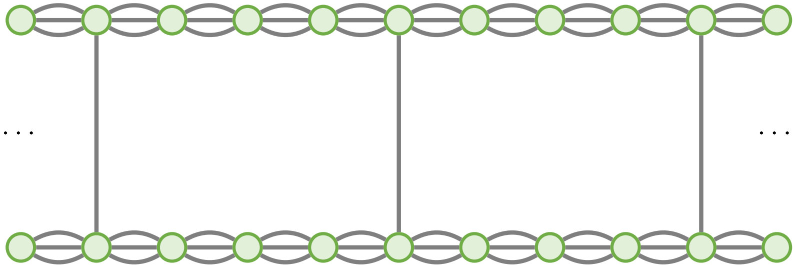

- Problem: Electrical connectivity is needed for the existence of spectrally thin trees. For any $e = (u, v) \in T$:

$$1 \geqslant \mathrm{Reff}_T(u, v) = e^\intercal L_T^- b_e \geqslant \frac{1}{\alpha} \cdot b_e^\intercal L_G^- b_e = \frac{1}{\alpha} \cdot \mathrm{Reff}_G(u, v).$$

Key Idea : Well-condition the graph spectrally without changing cuts much.

Well-Conditioning Scheme

- Add "graph" H to G ensuring

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$$|T(S, \bar{S})| = \mathbb{1}_S^\intercal L_T \mathbb{1}_S \leqslant \alpha \cdot \mathbb{1}_S^\intercal (L_G + L_H) \mathbb{1}_S = O(\alpha) \cdot |G(S, \bar{S})|$$

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- Goal: Find H that brings Reff down.
- Problem 1: How do we ensure T does not use any newly added edges?
- Problem 2: How do we certify H is $O(1)$-thin w.r.t. G?

Ensuring only original edges are picked . . .

Extension to Interlacing Families

[Harvey-Olver'14, Marcus-Spielman-Srivastava'14]

If for every edge e in a graph G

$$\mathrm{Reff}(e) \leqslant \alpha,$$

then G has an $O(\alpha)$-spectrally thin tree.

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Let F be a subset of edges in G. If for every $e \in F$,

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[on board . . .]

Ensuring cuts do not blow up . . .

Idea 1: Using Shortcuts

- If H can be routed over G with congestion $O(1)$, then for every S

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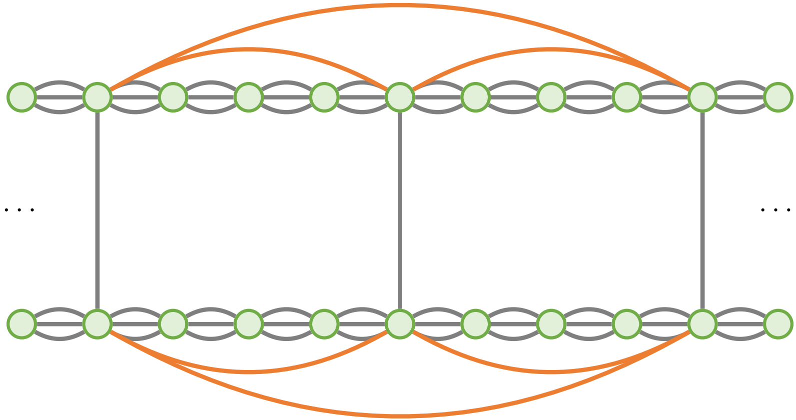

Idea 2: Check All Constraints

- Instead of $\mathbf{L}_H$, we can add any PSD matrix $\mathbf{D}$, as long as for all S

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- Just turn the problem into an exponential-sized semidefinite program:

$$\min_{D \succeq 0} \left\{ \max_{e \in G} \text{Reff}_D(e) \;\middle|\; \forall S : \mathbb{1}_S^\intercal D \mathbb{1}_S \leqslant \mathbb{1}_S^\intercal L_G \mathbb{1}_S \right\}$$

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- Pro: Can use duality to facilitate analysis.
- Con: Adds another obstacle to making the construction algorithmic.

Puzzle Interlude: Degree-thinness . . .

Degree-Thin Trees (Toy Example)

Suppose that we want a tree which is thin only in degree cuts, i.e.,

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- An expander!

[on board ...]

Do well-conditioners always exist?

- What is the worst possible answer to the convex program?

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Averages in Degree Cuts [A-Oveis Gharan'15]

For every k-edge-connected graph G there is a 1-thin matrix $D \succeq 0$ such that for every singleton S

$$\mathbb{E}[\mathrm{Reff}_D(e) \mid e \in G(S, \bar{S})] \leqslant \frac{(\log \log n)^{O(1)}}{k}.$$

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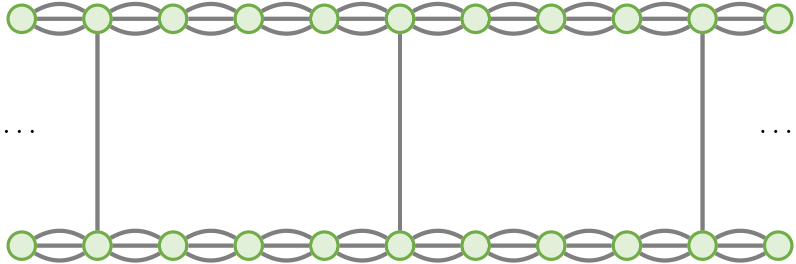
Informal Lemma

Every graph has weakly expanding induced subgraphs.

Plan: Contract this subgraph, and repeat to get a hierarchical decomposition. Lower average **Reff** in degree cuts of each expander simultaneously.

Example: Planar Graphs

If G is planar, there are vertices u and v connected by $\Omega(k)$ edges.



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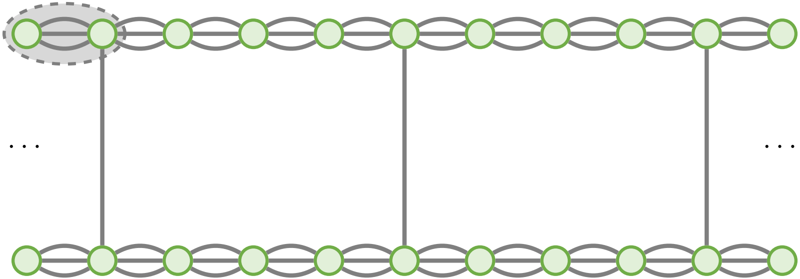

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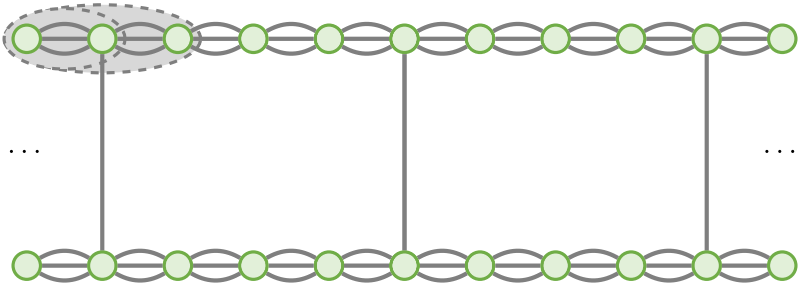

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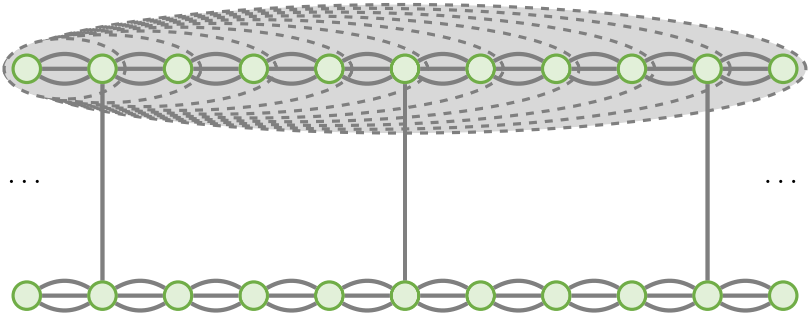

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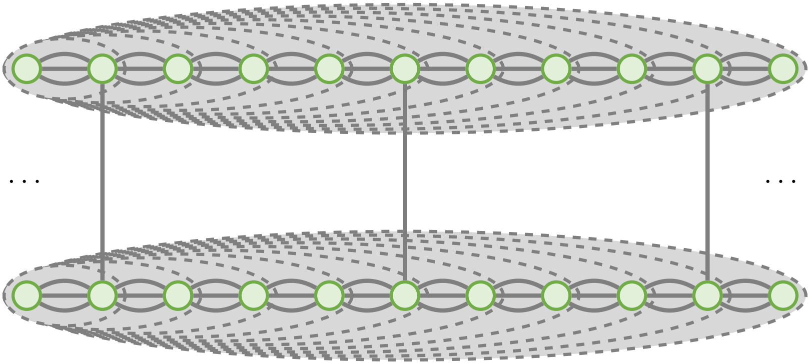

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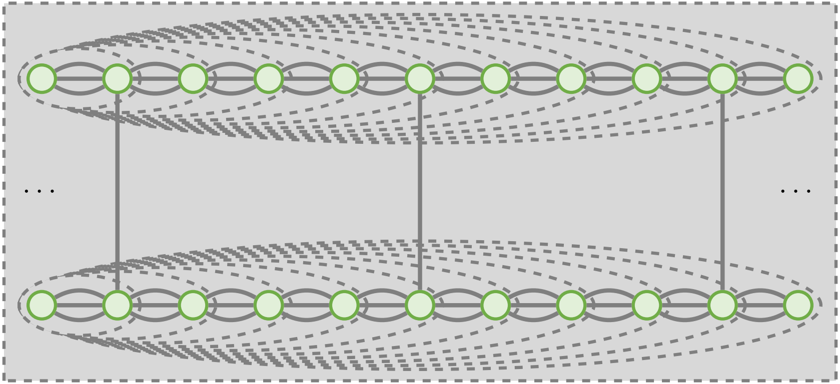

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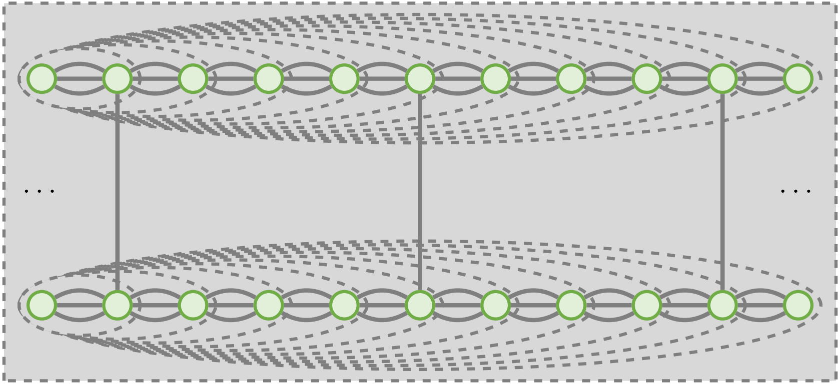

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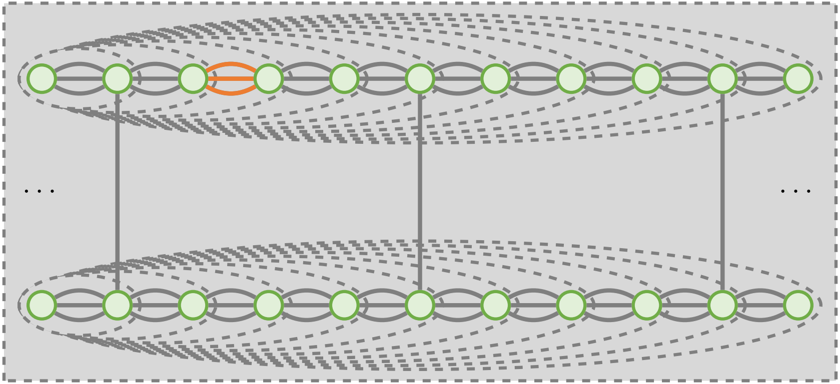

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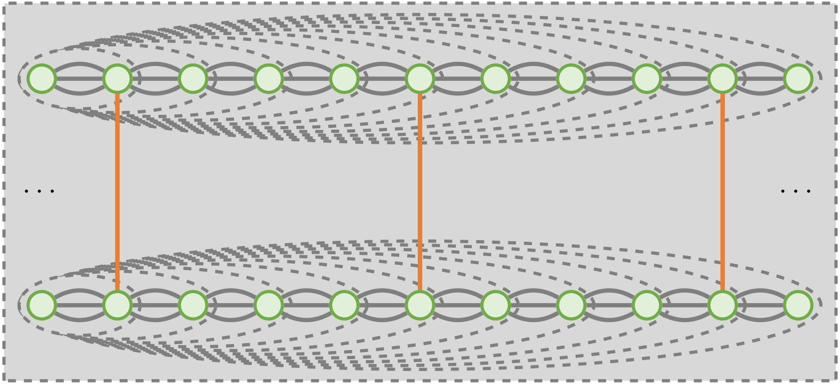

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- Key Observation: Expansion goes up by a constant factor after contracting.
- Repeat this $\log \log n$ times until expansion is $\Omega(1)$.

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Thank you!