A Tale of Santa Claus, Hypergraphs, and Matroids

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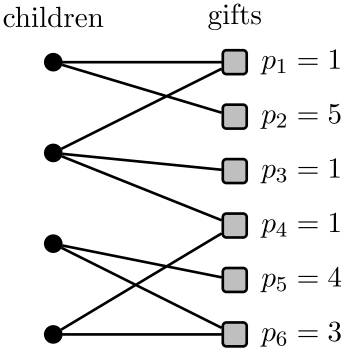

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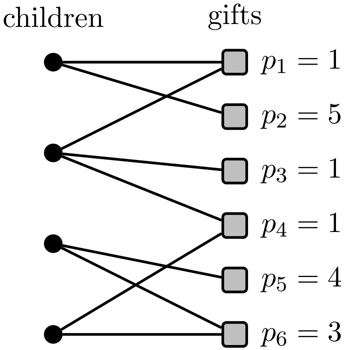

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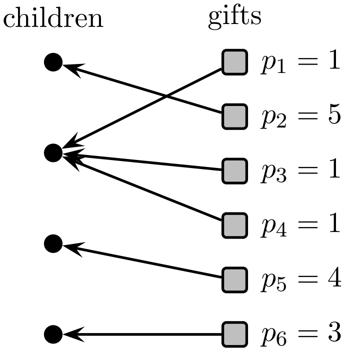

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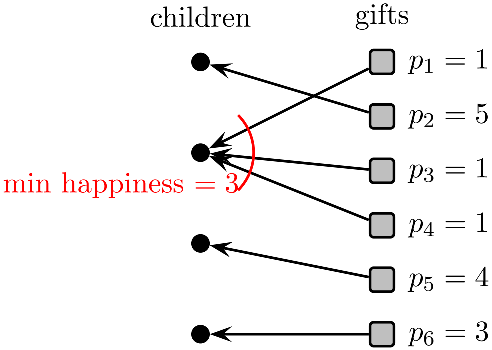

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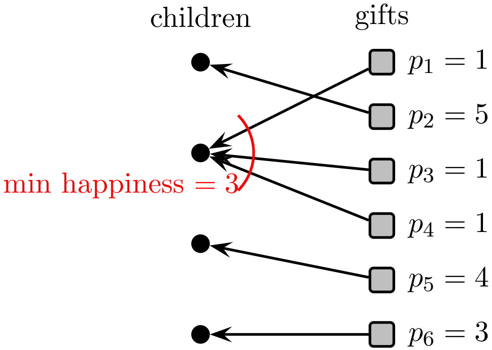

- Alternative name: **Restricted Max Min Fair Allocation**

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Here:

- An extension to **matroids**

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- **Example: Graphical matroid** $(E, \mathcal{I})$ ($G = (V, E)$ connected graph)
  - $\mathcal{I}$ = subset of forests
  - bases = spanning trees
  - base polytope = spanning tree polytope

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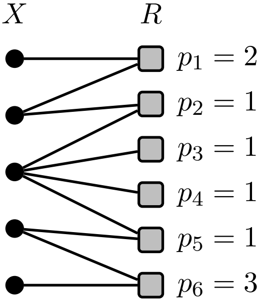

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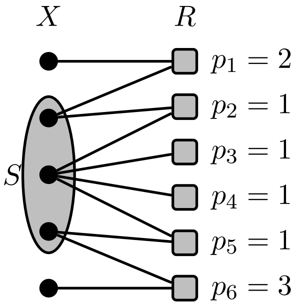

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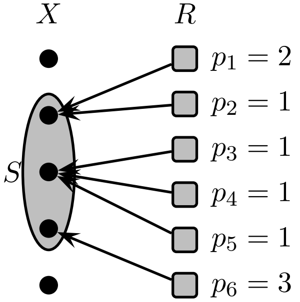

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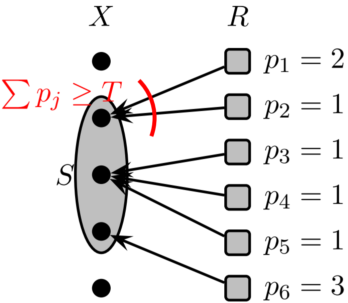

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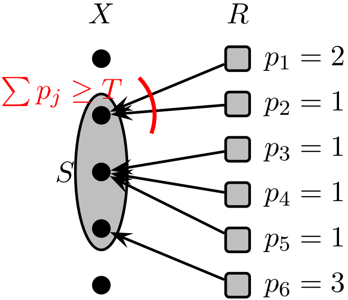

Linear program

$$
\begin{aligned}
x &\in P_{B(\mathcal{M})} \\
\sum_{j \in N(i)} p_j y_{ij} &\geq T \cdot x_i \; \forall i \in X \\
y(\delta(j)) &\leq 1 \; \forall j \in R \\
0 \leq y_{ij} &\leq x_i \; \forall (i,j) \in E
\end{aligned}
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Suppose LP feasible and $p_j = 1$. Then can find solution for **Matroid Max Min Fair Allocation** of value $(\frac{1}{3} - \varepsilon) \cdot T$ in **poly-time**.

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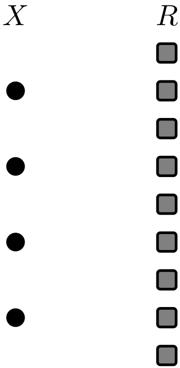
- [Cheng-Mao '18] obtain $(6 + \varepsilon)$-apx by directly modifying [AKS'15]

General approach

- **Assumptions:** $p_j = 1$ & LP is feasible for parameter T



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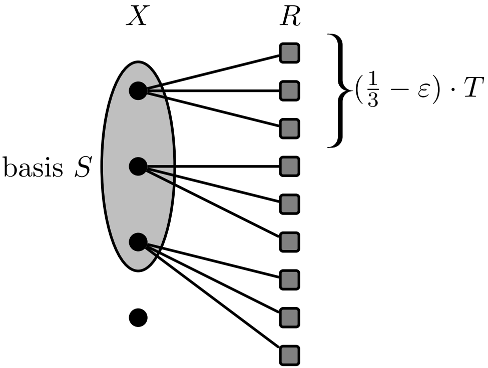

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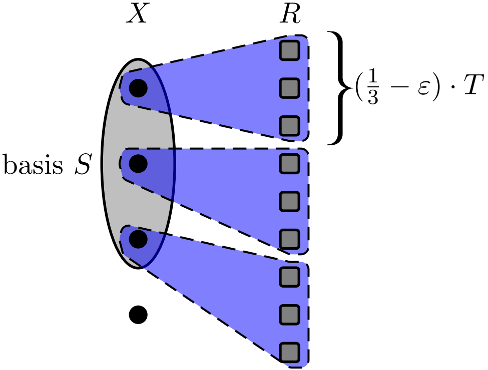

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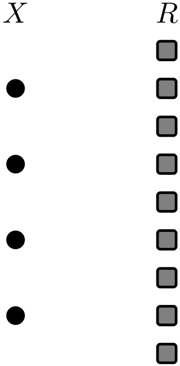

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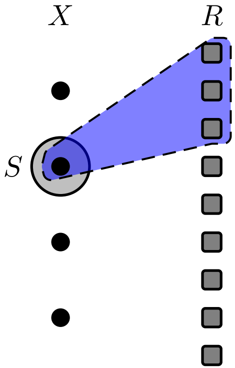

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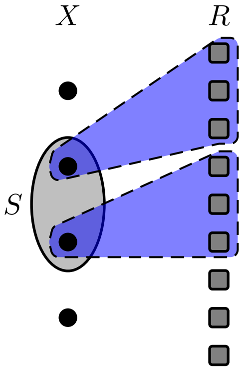

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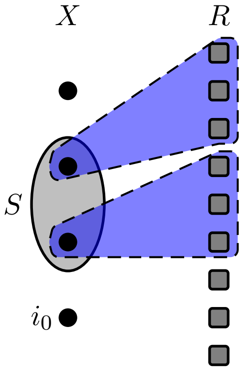

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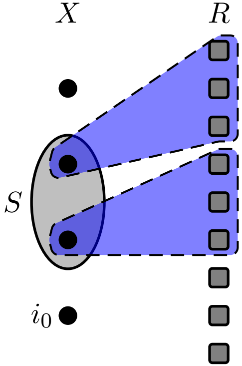

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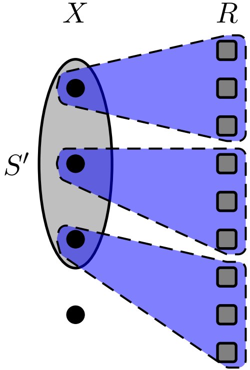

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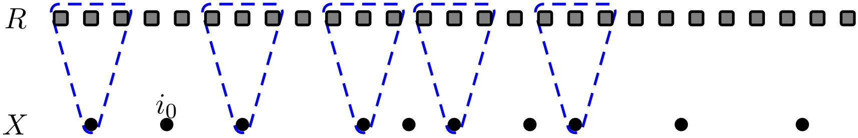

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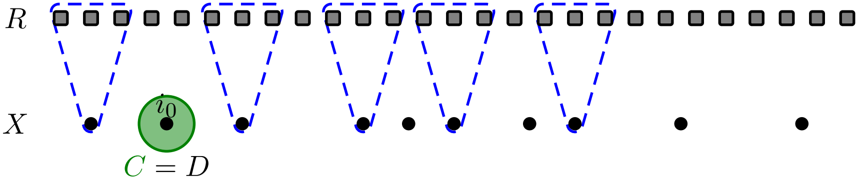

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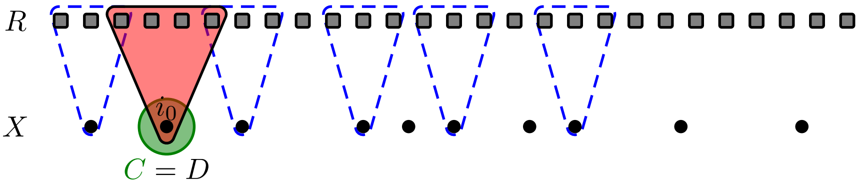

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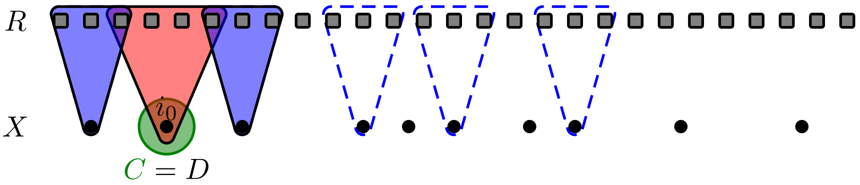

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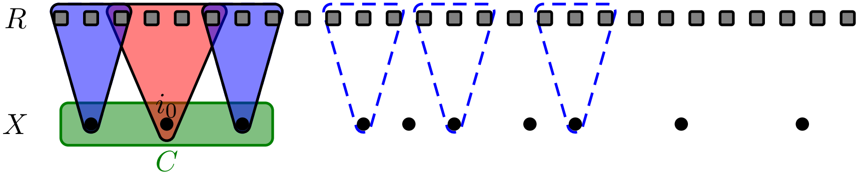

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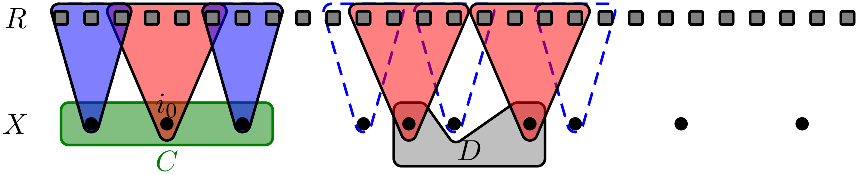

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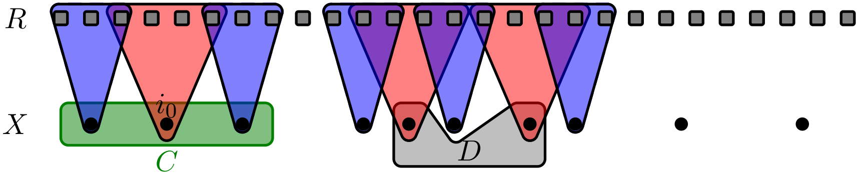

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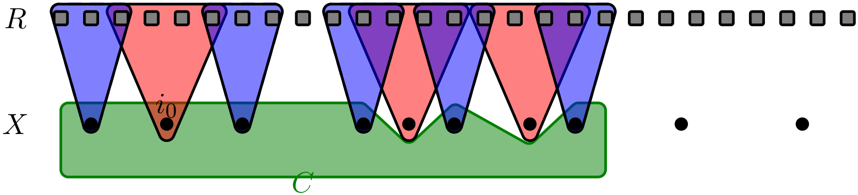

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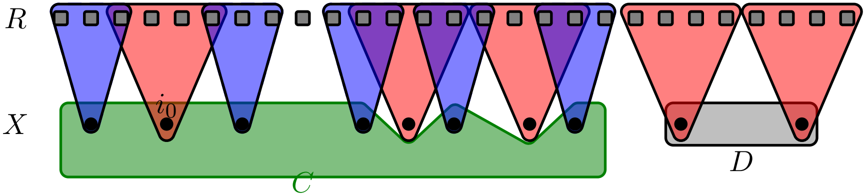

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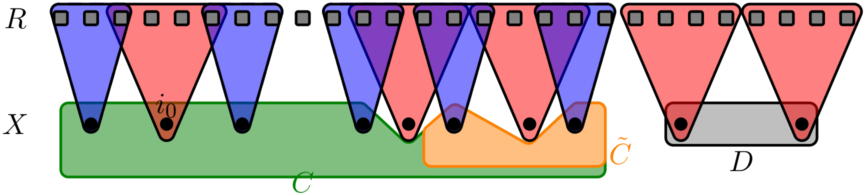

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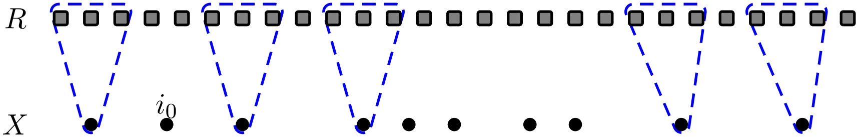

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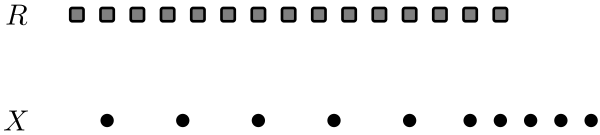

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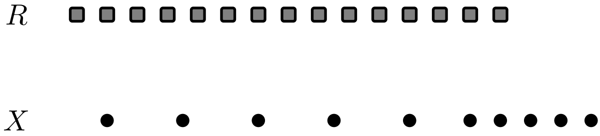

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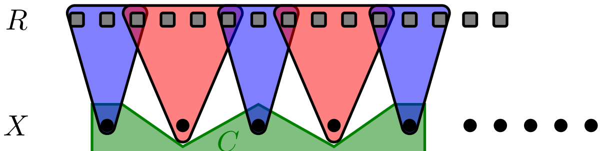

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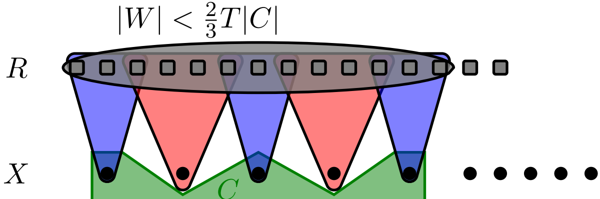

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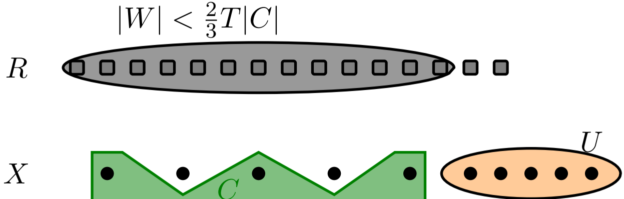

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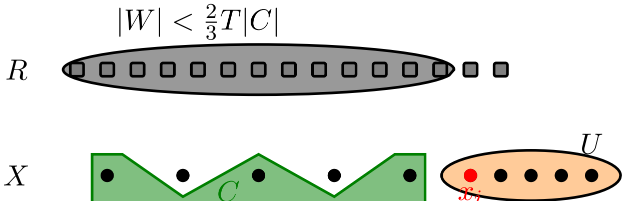

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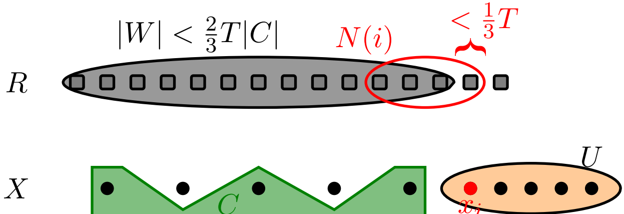

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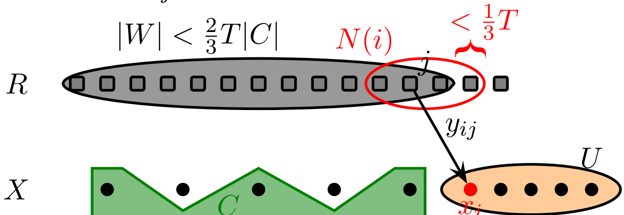

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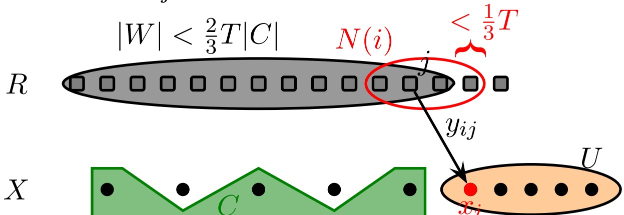

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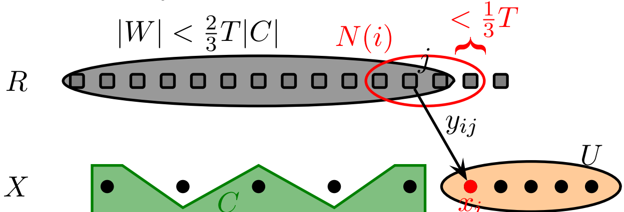

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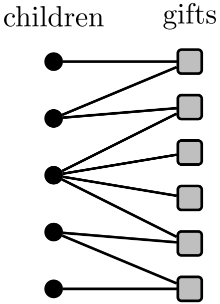

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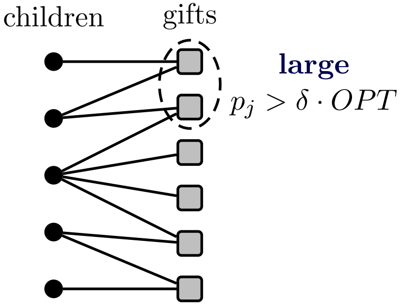

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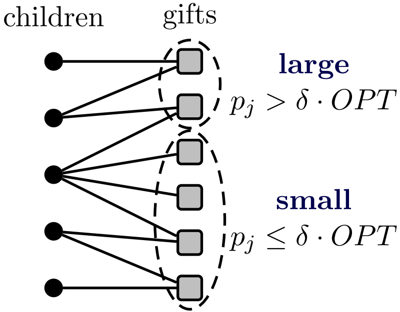

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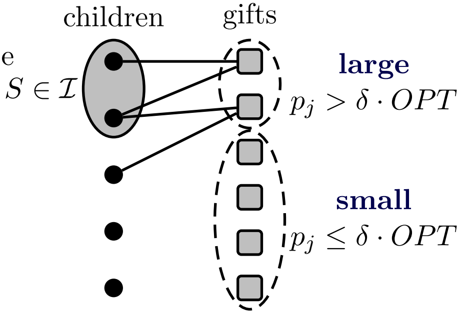

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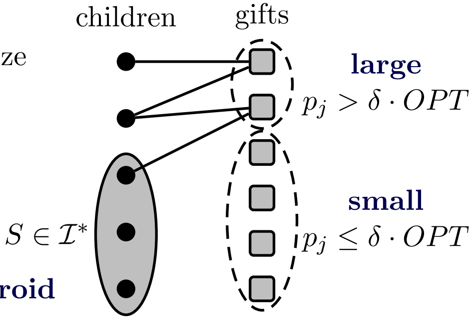

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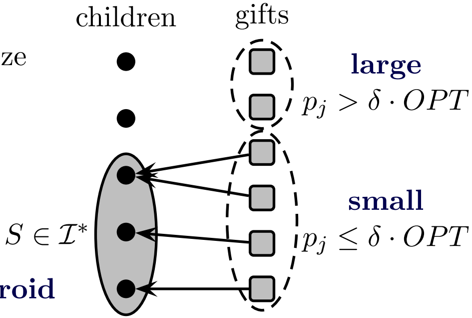

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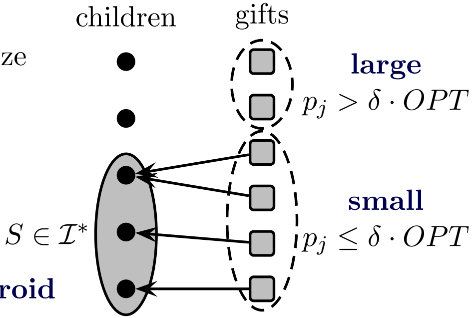

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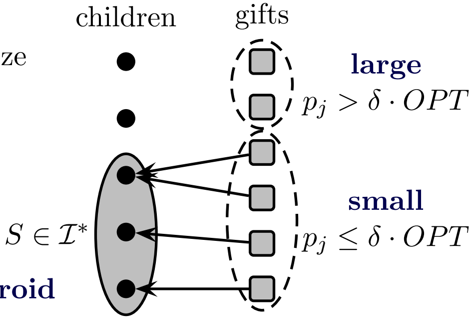

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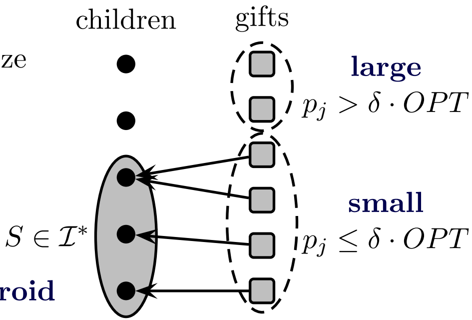

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