Averages of Laplace eigenfunctions

Joint works with J.Galkowski and J.Toth

Problem

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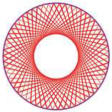

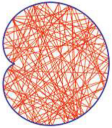

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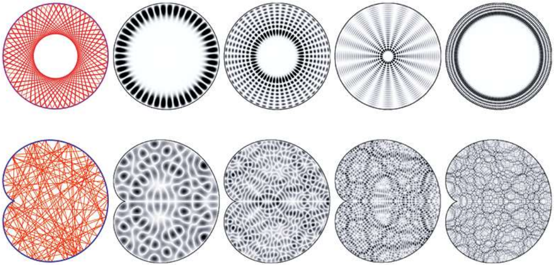

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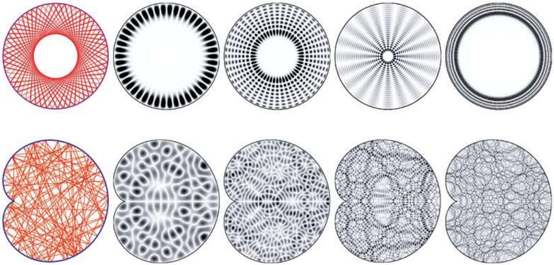

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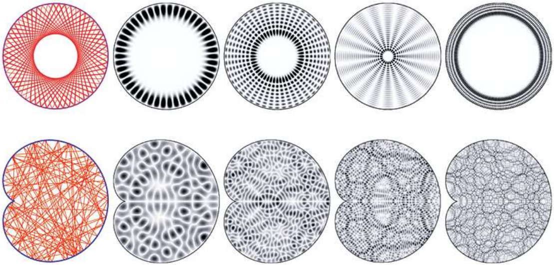

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Obs. If $H = \{x\}$ we get info on $\phi_\lambda(x)$.

What's known

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$\int_H \phi_\lambda \, d\sigma_H = o(1)$	Chen-Sogge'15 *M* sect.curv<0 *H* geodesic	$\int_H \phi_\lambda \, d\sigma_H = o(\lambda^{\frac{k-1}{2}})$	Wyman'17 $\sigma_{SN^*H}(\mathcal{L}_H) = 0$
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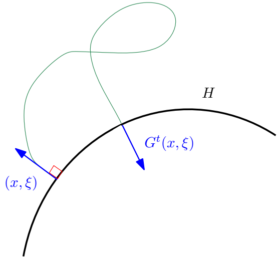
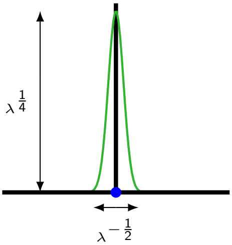


$\mathcal{L}_H = \{(x,\xi) \in SN^*H \text{ that loop back to } SN^*H\}$

Gaussian beam heuristics

Profile across a gaussian beam



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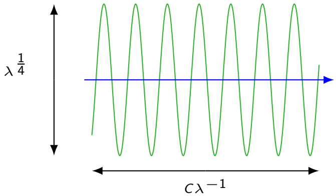

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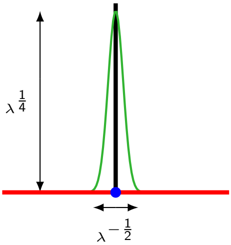

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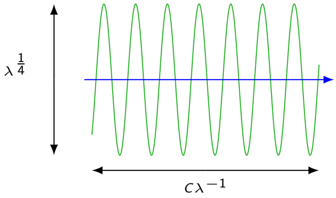

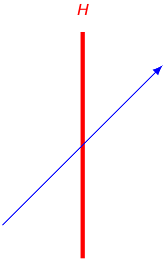

$$\|\phi_\lambda\|_{L^2} = 1$$

$$\int_H \phi_\lambda \, d\sigma_H = O(\lambda^{-\infty})$$

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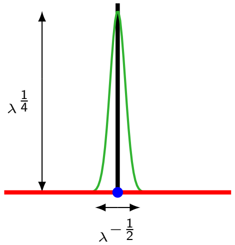

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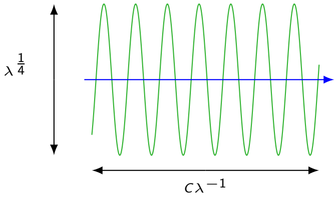

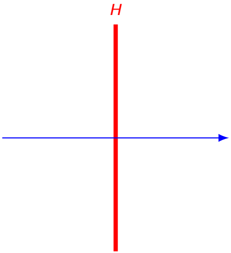

$$\|\phi_\lambda\|_{L^2} = 1$$

$$\int_H \phi_\lambda d\sigma_H \sim c\lambda^{-\frac{1}{4}}$$

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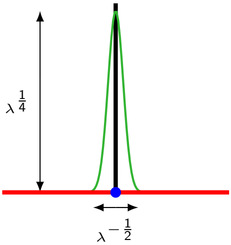

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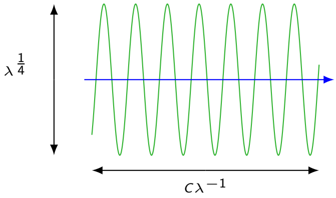

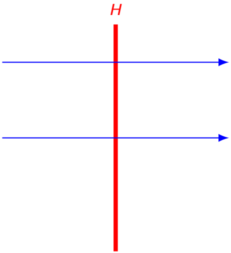

$$\|\phi_\lambda\|_{L^2} = \sqrt{2}$$

$$\int_H \phi_\lambda d\sigma_H \sim 2\, c\lambda^{-\frac{1}{4}}$$

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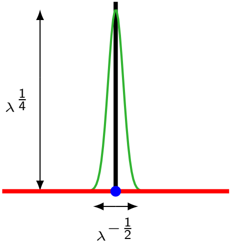

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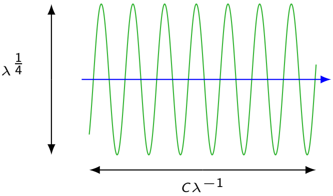

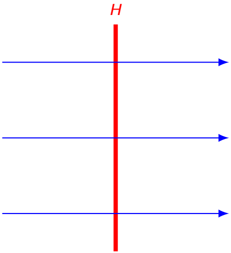

$$\|\phi_\lambda\|_{L^2} = \sqrt{3}$$

$$\int_H \phi_\lambda \, d\sigma_H \sim 3\, c\lambda^{-\frac{1}{4}}$$

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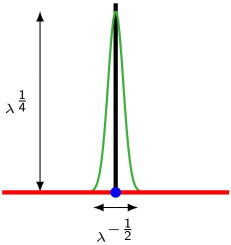

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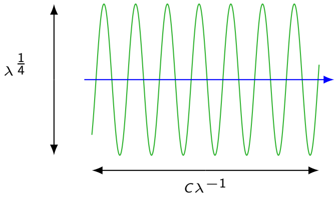

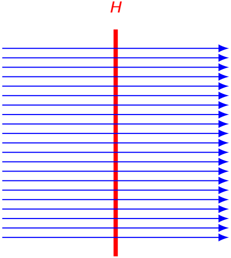

$$\|\phi_\lambda\|_{L^2} = \sqrt{\lambda^{\frac{1}{2}}}$$

$$\int_H \phi_\lambda \, d\sigma_H \sim \underbrace{\lambda^{\frac{1}{2}} c\lambda^{-\frac{1}{4}}}_{c\,\lambda^{\frac{1}{4}}}$$

Defect measures

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- $\{\phi_\lambda\}$ **is quantum ergodic:** μ is the Liouville measure on S^*M.

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Always true if $\{\phi_\lambda\}$ is a Quantum Ergodic sequence.

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Theorem (C-Galkowski. Key estimate)

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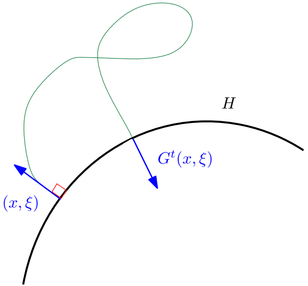
$\Longrightarrow$ If $\{\phi_\lambda\}$ has maximal averages, then μ_H and σ_{SN^*H} are **not** mutually singular.

- Torus example: $f = 1$ (average is saturated)
- Gaussian Beam: $f = 0$ (average goes to 0)

Recurrent co-normal directions



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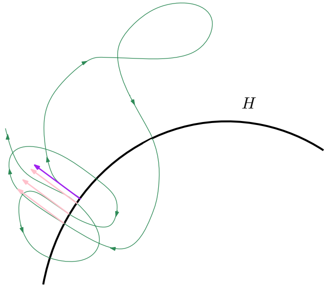

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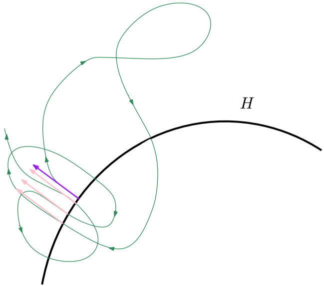
Recurrent co-normal directions

Theorem (C-Galkowski)

$$\mu_H(\mathcal{R}_H) = \mu_H(SN^*H).$$



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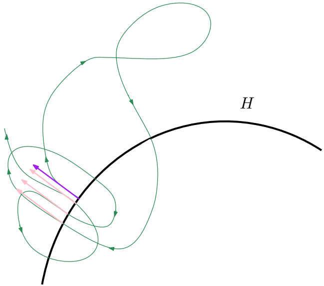

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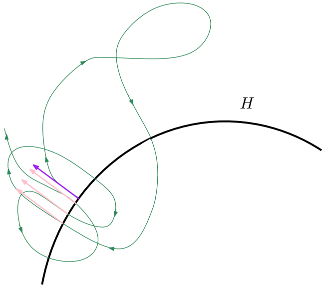
$$\mu_H(\mathcal{R}_H) = \mu_H(SN^*H).$$

Theorem (C-Galkowski)

If $\sigma_{SN^*H}(\mathcal{R}_H) = 0$*, then for every sequence* $\{\phi_\lambda\}$

$$\int_H \phi_\lambda d\sigma_H = o(\lambda^{\frac{k-1}{2}}).$$

Recurrent co-normal directions



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Theorem (C-Galkowski)

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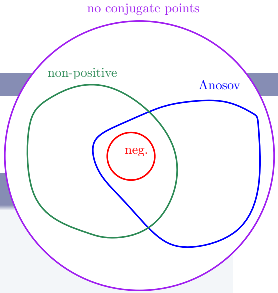
$$\int_H \phi_\lambda d\sigma_H = o(\lambda^{\frac{k-1}{2}}).$$

Remember $\mu_H = f\sigma_{SN^*H} + \lambda_H$, so $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ implies $\mu_H \perp f\sigma_{SN^*H}$.

Submanifolds with $\sigma_{SN^*H}(\mathcal{R}_H) = 0$



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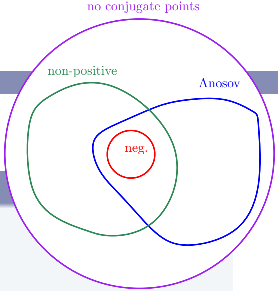

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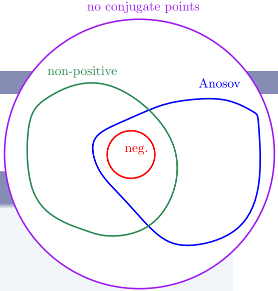

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- *(M, g) has constant negative curvature and H is any submanifold.*

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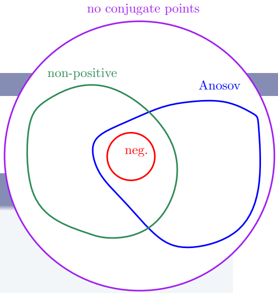


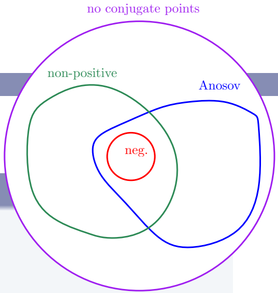
Theorem (C-Galkowski)

The following setting imply

$$\sigma_{SN^*H}(\mathcal{R}_H) = 0.$$

- (M, g) *has constant negative curvature and H is any submanifold.*
- (M, g) *is a surface with Anosov geodesic flow and H is any curve.*

Submanifolds with $\sigma_{SN^*H}(\mathcal{R}_H) = 0$



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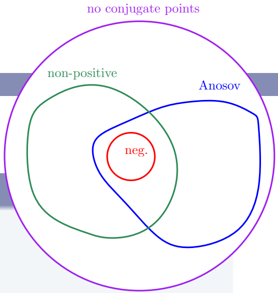

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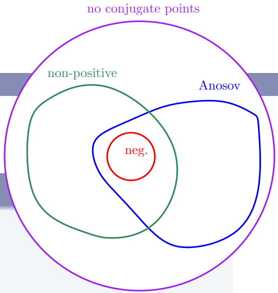

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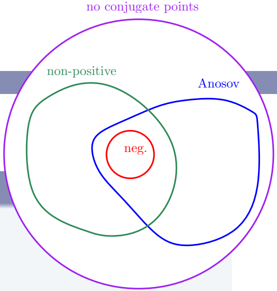

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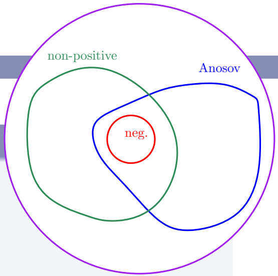
All these imply

$$\int_H \phi_\lambda d\sigma_H = o(\lambda^{\frac{k-1}{2}}).$$

Logarithmic improvements



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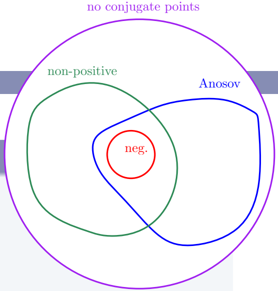

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In addition, if $x \in M$ is **not** *self-conjugate with maximal multiplicity,*

$$\left\|\phi_\lambda\right\|_{L^\infty(B(x,\lambda^{-\delta}))} = O\left(\frac{\lambda^{\frac{n-1}{2}}}{\sqrt{\log \lambda}}\right).$$

Thank you!

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