Jones modes in Lipschitz domains

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Fluid-solid interaction

Hemholtz equation for p:

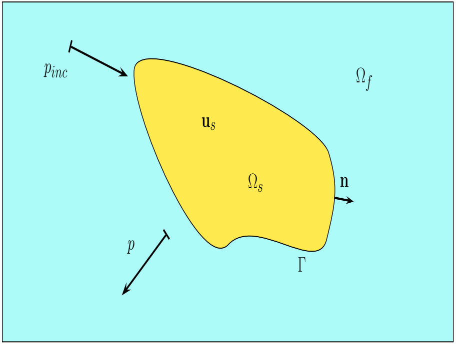
$$\Delta p + (w/c)^2 p = 0,$$

Linear elasticity: $\mu_s > 0$, $\lambda_s + \left(\frac{2}{d}\right)\mu_s > 0$

$$\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_s) + w^2 \rho_s \tilde{\mathbf{u}}_s = \mathbf{0}.$$

BC's on the interface:

$$\boldsymbol{\sigma}(\mathbf{u}_s)\mathbf{n} = -(p + p_{inc})\mathbf{n},$$
$$w^2 \rho_f \mathbf{u}_s \cdot \mathbf{n} = \nabla(p + p_{inc}) \cdot \mathbf{n}.$$



Condition at infinity:

$$\frac{\partial p}{\partial r} - i(w/c)p = o(1/r), \qquad r := \|\mathbf{x}\|.$$

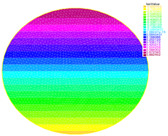
[2] Hsiao, Kleinman and Roach 2000; Hsiao, Xu and Yin 2017

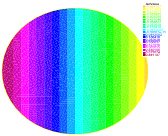
Non-uniqueness

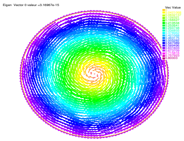
Lemma

If $(\mathbf{u}_s, p)$ *solves the time harmonic fluid-solid interaction problem then* $(\mathbf{u}_s + \mathbf{u}_0, p)$ *also solves this problem, with* $\mathbf{u}_0$ *a non-zero solution of*

$$\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_0) + \rho_s w^2 \mathbf{u}_0 = \mathbf{0}, \ \text{in } \Omega_s, \quad \boldsymbol{\sigma}(\mathbf{u}_0)\mathbf{n} = \mathbf{0}, \quad \mathbf{u}_0 \cdot \mathbf{n} = 0, \ \text{on } \Gamma.$$







Note

- *Condition on shear along the interface;*
- *Robin condition in an "artificial" boundary away from the solid;*
- *no eigenpairs for* C^∞ *domains in* $\mathbb{R}^3$.

HKR, 2000; Gatica et al., 2009; Barucq et al., 2014; T. Hargé, 1990

The EV problem: Jones modes

This problem is not uniquely solvable when $w^2\rho_s$ is an eigenvalue of

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_s) + w^2\rho_s\mathbf{u}_s &= \mathbf{0}, \quad \text{in } \Omega_s, \\ \boldsymbol{\sigma}(\mathbf{u}_s)\mathbf{n} = \mathbf{0}, \quad \mathbf{u}_s\cdot\mathbf{n} &= 0, \quad \text{on } \partial\Omega_s. \end{aligned}$$

A non-trivial solution $\mathbf{u}_s$ for some w^2 is called a *Jones mode*.

Note

Over-determined EV:

Elasticity equation + Traction condition + extra constraint on $\mathbf{u}_s$.

[2] Jones et al., 1983 and 1984

shear and compression modes

$$s\text{−waves: } \operatorname{div} \mathbf{u}_s = 0 \qquad p\text{−waves: } \operatorname{rot} \mathbf{u}_s = \mathbf{0}$$

On $[0, a] \times [0, b]$,

$$w_{mn}^2 := \begin{cases} \left(\frac{\pi^2 \mu}{\rho}\right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \\ \left(\frac{\pi^2(\lambda+2\mu)}{\rho}\right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \end{cases}$$

with eigenfunctions:

$$\mathbf{u}_{mn} := \begin{cases} an \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\hat{i} - bm\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\hat{j}, \\ bm \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\hat{i} + an\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\hat{j}, \end{cases}$$

Weak formulation (on Ω_s)

Consider $\mathbf{H}^1(\Omega) := H^1(\Omega) \times H^1(\Omega)$ and define

$$\mathbf{H} := \left\{ \mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \right\}.$$

FORMULATION: find $(w^2, \mathbf{u}) \in \mathbb{C} \times \mathbf{H}$ such that

$$a(\mathbf{u}, \mathbf{v}) = \rho w^2 (\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{H},$$

where $a(\mathbf{u}, \mathbf{v}) := \mu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\lambda + \mu)(\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})$ or $(\boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\epsilon}(\mathbf{v}))$.

Note

- $a(\mathbf{u}, \mathbf{v}) \leq (\lambda + \mu)\|\mathbf{u}\|_1 \|\mathbf{v}\|_1$;
- *Rayleigh quotient + properties of* $a(\cdot, \cdot)$ *and* $(\cdot, \cdot) \Rightarrow w^2 \geq 0$.

Ellipticity of a

We can get

$$a(\mathbf{u},\mathbf{u}) \geq \min\left\{2\mu, d(\lambda + (2/d)\mu)\right\}\|\boldsymbol{\epsilon}(\mathbf{u})\|_0^2, \quad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega).$$

Ellipticity of a

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$$a(\mathbf{u},\mathbf{u}) \geq \min\left\{2\mu, d(\lambda + (2/d)\mu)\right\}\|\epsilon(\mathbf{u})\|_0^2, \quad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega).$$

Theorem (Bauer 2016, Domínguez 2018)

Let Ω be a non-axisymmetric bounded and Lipschitz domain in $\mathbb{R}^d$. Then, there is a positive constant $C > 0$ such that

$$\|\epsilon(\mathbf{u})\|_0 \geq C\|\mathbf{u}\|_1, \quad \forall\, \mathbf{u} \in \mathbf{H}.$$

Existence of eigenpairs

Lemma

Under the same assumptions for Ω, there is a constant $C > 0$ such that

$$a(\mathbf{u}, \mathbf{u}) \geq c\|\mathbf{u}\|_1^2, \quad \forall\, \mathbf{u} \in \mathbf{H}.$$

The corresponding solution operator T is then

- linear and bounded with $\|T\|_{\mathbf{H}'} = \frac{\rho}{C}$;
- compact from $\mathbf{H}$ to itself;
- self-adjoint w.r.t. $a(\cdot, \cdot)$.
- Spectral Theorem $\Rightarrow$ eigenpairs $\{w_n\}$ and $\{\mathbf{u}_n\}$ with $w_n \to +\infty$.

Rigid motions

Lemma

$w^2 = 0$ is an eigenvalue with:

(i) *a pure translation as eigenfunction if $\partial\Omega$ consists of two parallel planes;*

(ii) *a pure rotation $\mathbf{u}_0$ as eigenfunction if Ω is axisymmetric about the axis of rotation of $\mathbf{u}_0$.*

Shifted formulation: find $(\mathbf{u}, w^2) \in \mathbf{H} \times \mathbb{R}$ such that

$$\tilde{a}(\mathbf{u},\mathbf{v}) := a(\mathbf{u},\mathbf{v}) + \rho(\mathbf{u},\mathbf{v}) = \rho(w^2+1)(\mathbf{u},\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{H},$$

- $\tilde{a}(\mathbf{u},\mathbf{u}) \geq \min\{\mu, \rho\}\|\mathbf{u}\|_1^2$;
- the corresponding solution operator $\tilde{T}$ is well-defined, compact and self-adjoint;

Discrete scheme

Let $\mathbf{H}_h \subseteq \mathbf{H}$ (Lagrange elements): find $\mathbf{u}_h \in \mathbf{H}_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = \rho\kappa_h(\mathbf{u}_h, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h,$$

with $\kappa_h := w_h^2$ or $w_h^2 + 1$.

- a is $\mathbf{H}_h$-elliptic;
- a discrete solution operator T_h (cf. $\tilde{T}_h$) is well defined;
- error bound for evs:

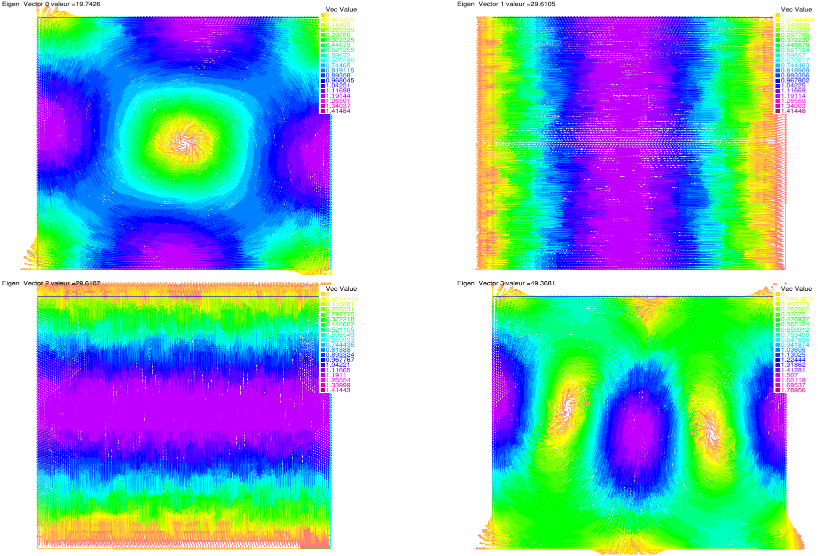
$$\frac{|\kappa - \kappa_h|}{\kappa} \leq Ch^{2(t-1)}, \quad t > 1.$$

- operator approximation:

$$\|T - T_h\| \leq ch^{t-1}.$$

[2] Babuška and Osborn, 1991.

Square ($\mu = \lambda = 1$)



Square (contd.)

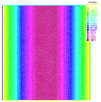
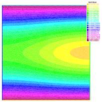
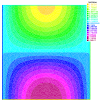
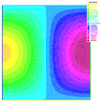
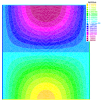
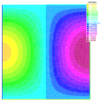
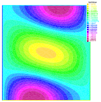
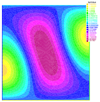
j	w^2	w^2/π^2	$\|\mathrm{div}\,\mathbf{u}\|_0^2$	$\|\mathrm{rot}\,\mathbf{u}\|_0^2$	$x-$component	$y-$component
1	19.74	2.000	9.870	0.0002633		
2	19.74	2.000	9.870	0.0001704		
3	19.74	2.000	5.883e-05	19.72		
4	39.48	4.000	19.74	0.0005061		

Table: Unit square with parameters $\mu = \rho = 1$, $\lambda = 0$.

Square (contd.)

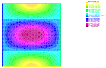
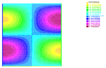
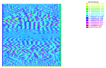
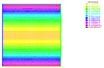
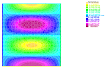
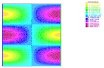
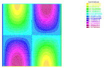
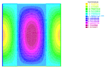
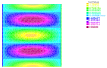
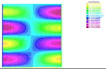
j	ν_j	ν_j/π^2	$\|\mathrm{div}\,\mathbf{u}\|_0^2$	$\|\mathrm{rot}\,\mathbf{u}\|_0^2$	$x-$component	$y-$component
1	12.34	1.250	1.231e-07	12.27		
2	19.74	2.000	9.515e-07	19.69		
3	29.61	3.000	2.467	2.271e-05		
4	32.08	3.250	2.74e-06	32.05		
5	41.95	4.250	2.223e-06	41.62		

Table: 2X1 rectangle with parameters $\mu = \rho = 1, \lambda = 10$.

Square (contd.)

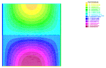
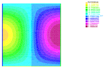
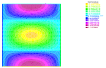
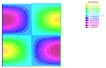
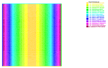
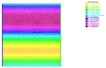
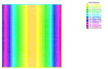
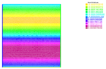
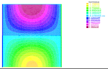
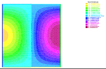
j	ν_j	ν_j/π^2	$\|\text{div}\,\mathbf{u}\|_0^2$	$\|\text{rot}\,\mathbf{u}\|_0^2$	x−component	y−component
1	51.82	5.25	2.467	2.271e-05		
2	123.4	12.5	2.271e-07	12.27		
3	197.4	20	1.757e-06	19.69		
4	207.3	21	9.869	0.0004004		
5	207.3	21	9.87	0.000243		

Table: 2X1 rectangle with parameters $\mu = 10, \lambda = \rho = 1$.

Triangle

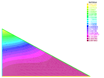
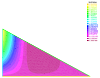
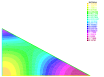
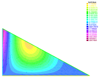
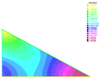
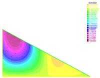
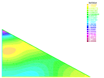
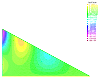
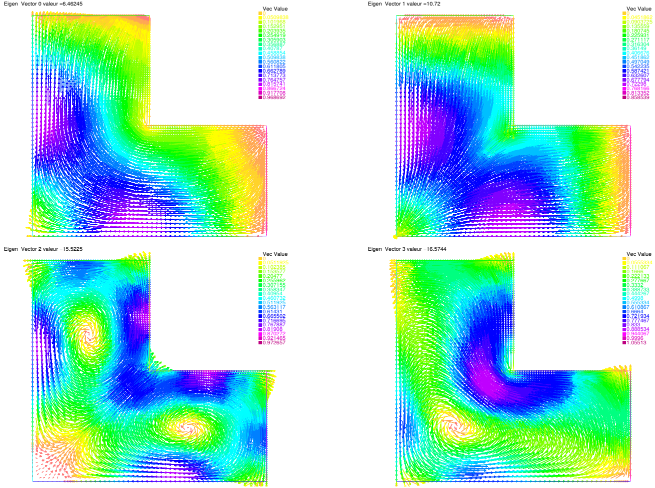
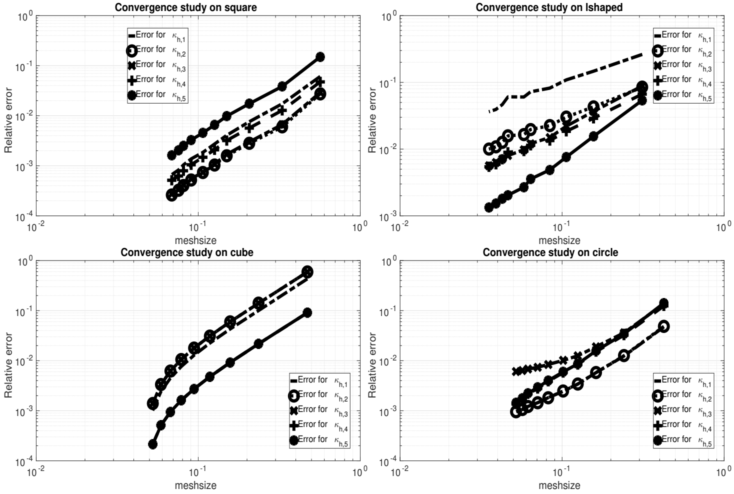
j	w^2	w^2/π^2	$\|\text{div}\,\mathbf{u}\|_0^2$	$\|\text{rot}\,\mathbf{u}\|_0^2$	$x-$component	$y-$component
1	4.6563	0.4718	0.7007	24.36		
2	8.3125	0.8422	0.4333	14.42		
3	11.84674	1.200	2.527	4.15		
4	21.0647	2.134	1.640	75.96		

Table: Isosceles triangle of vertices $(0,0)$, $(2,0)$ and $(1,2)$ with parameters $\lambda = \mu = \rho = 1$.

L-shape, $\rho = \mu = \lambda = 1$



CV properties on polyhedron



Conclusions and future work

Conclusions:

- FEM provides a reliable scheme to approximate Jones modes on polyhedral domains;
- the extra constraint makes the problem "*domain dependent*".

Future work:

- Does this scheme work for more general smooth domains?
- A posteriori error analysis would help to improve the convergence for computations on non-convex domains.

Thanks!

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