Adaptive approximation of eigenproblems: multiple eigenvalues and clusters

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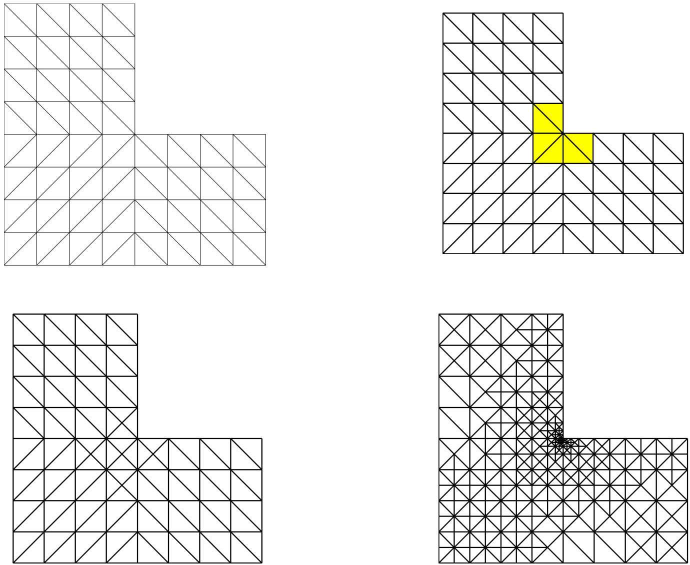
Banff, July 1-6, 2018

Based on joint works with D. Boffi, R. Durán, D. Gallistl, L. Gastaldi

Outline

1. A priori estimates for multiple eigenvalues
2. A posteriori estimates for multiple eigenvalues and clusters
3. Adaptive strategy for eigenvalue problems in mixed form (cluster-robust)

Uniform vs. adaptive mesh refinement



Adaptive FEM for eigenvalue problems

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

A posteriori error estimator

Set $e_h = u - u_h$, build

an error estimator $\eta = \eta(\lambda_h, u_h) = \left(\sum_{K\in\mathcal{T}_h} \eta_K^2(\lambda_h, u_h)\right)^{1/2}$ such that

- $\|e_h\|_1 \le C_1\eta$ (Reliability)
- $\eta_K \le C_2\|e_h\|_{1,K^*}$ (Efficiency)

with C_1, C_2 constants independent of h

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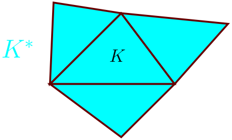

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- $\eta_K \leq C_2 \|e_h\|_{1,K^*}$ (Efficiency)

with C_1, C_2 constants independent of h

Remark

If the estimator is reliable, then it also holds $|\lambda - \lambda_h| \leq C_\lambda \eta^2$

Standard Adaptive Finite Element Method for eigenvalue problems

Input

Parameter $\theta \in (0, 1]$ and initial triangulation $\mathcal{T}_0$

SOLVE, ESTIMATE, MARK, REFINE

Solve: Compute discrete solution (λ_ℓ, u_ℓ) on $\mathcal{T}_\ell$

Estimate: Compute local contributions of the error estimator $\{\eta_\ell^2(T)\}_{T\in\mathcal{T}_\ell}$

Mark: Choose minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ such that $\theta\eta_\ell^2(\mathcal{T}_\ell) \le \eta_\ell^2(\mathcal{M}_\ell) \quad (0 < \theta \le 1)$

Refine: Generate new triangulation as the smallest refinement of $\mathcal{T}_\ell$ satisfying $\mathcal{M}_\ell \cap \mathcal{T}_{\ell+1} = \emptyset$

Output

Sequence of meshes $\{\mathcal{T}_\ell\}$, sol.'s $\{(\lambda_\ell, u_\ell)\}$, indicators $\{\eta_\ell(\mathcal{T}_\ell)\}$

Clusters of eigenvalues

When multiple eigenvalues are present, a posteriori error indicators should be based simultaneously on all eigenfunctions belonging to the corresponding eigenspace

⟨Solin–Giani, '12⟩
⟨Boffi–Durán–G.–Gastaldi, '17⟩

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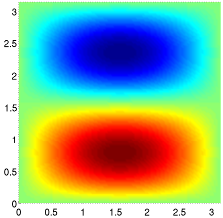
It is now recognized that an adaptive scheme for the approximation of eigenvalue problems should be designed taking into account error indicators based on all eigenmodes belonging to clusters of eigenvalues

⟨Gallistl '14⟩

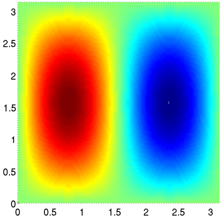
Example of approximation of multiple eigenvalues

5



$\sin(x)\sin(2y)$

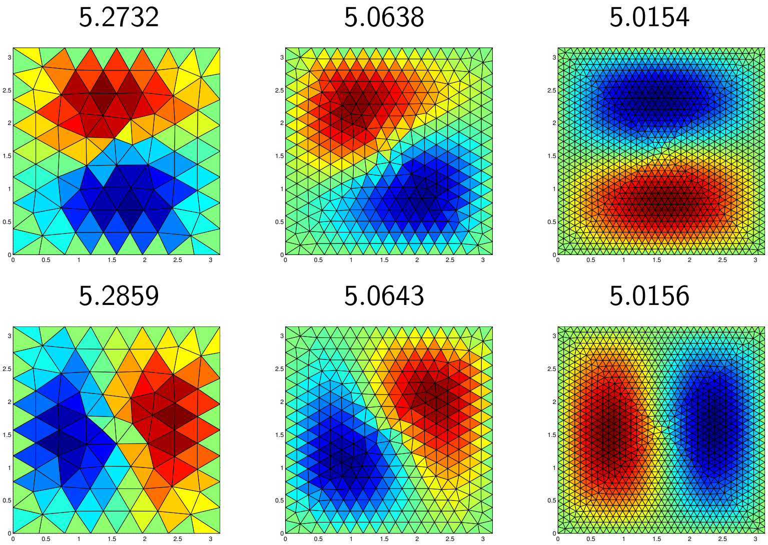
5



$\sin(2x)\sin(y)$

Multiple eigenvalues (cont'ed)

P1 approximation of multiple eigenvalue on unstructured meshes



A priori error analysis

⟨Babuška–Osborn, '91, Boffi '10⟩

Variational setting ($V \subset \mathcal{H}$ compact, $a : V \times V \to \mathbb{R}$ symmetric)

$$\lambda \in \mathbb{R},\ u \in V \qquad a(u, v) = \lambda(u, v)_{\mathcal{H}} \qquad \forall v \in V$$

$$\lambda_h \in \mathbb{R},\ u_h \in V_h \qquad a(u_h, v) = \lambda_h(u_h, v)_{\mathcal{H}} \qquad \forall v \in V_h$$

$E(\lambda)$ eigenspace associated with λ

$$\varepsilon(\lambda) = \sup_{\substack{u \in E(\lambda) \\ \|u\|=1}} \inf_{v_h \in V_h} \|u - v_h\|_V$$

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Simple eigenvalue

$$\sup_{\substack{u\in E(\lambda)\\ \|u\|=1}} \inf_{u_h\in E_h(\lambda_h)} \|u - u_h\|_V \le C\varepsilon(\lambda)$$

$$|\lambda - \lambda_h| \le C\varepsilon(\lambda)^2$$

A priori analysis (multiple eigenvalues)

λ of multiplicity m approximated by $\{\lambda_{i,h}\}$, $i = 1, \dots m$

$\hat{E}_h(\hat{\lambda}) = E_h(\lambda_{1,h}) \oplus \cdots \oplus E_h(\lambda_{m,h})$

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Standard estimates involve the angle between subspaces (gap)

$$\sup_{\substack{u \in E(\lambda) \\ \|u\|=1}} \inf_{\hat{u}_h \in \hat{E}_h(\hat{\lambda})} \|u - \hat{u}_h\|_V \le C\varepsilon(\lambda)$$

$$|\lambda - \lambda_{i,h}| \le C\varepsilon(\lambda)^2 \qquad i = 1, \dots, m$$

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$$|\lambda - \lambda_{i,h}| \le C\varepsilon(\lambda)^2 \qquad i = 1, \dots, m$$

More refined estimates

In case of eigenspace $E(\lambda)$ containing eigenfunctions of different regularity, the error estimates can be improved to reflect this aspect

⟨Knyazev–Osborn, '06⟩

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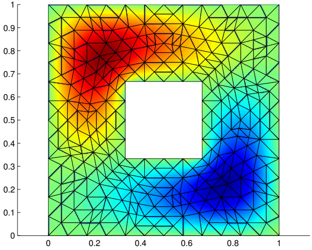
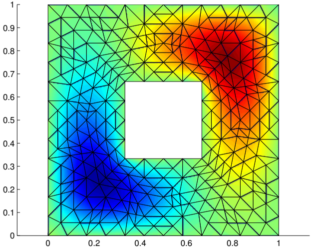
A posteriori error estimates: the square ring

⟨Solin–Giani, '12⟩

λ_2



$\lambda_3 = \lambda_2$



A posteriori error estimates: the square ring

⟨Solin–Giani, '12⟩

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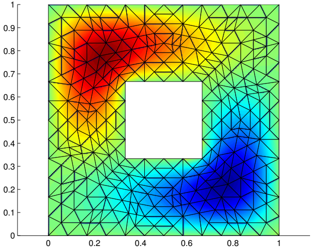

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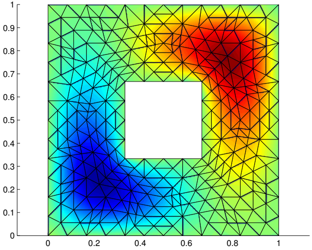

Question

What is the best adaptive strategy for the approximation of the multiple eigenvalue?

Finite element formulation

⟨Dari–Durán–Padra, '12⟩
⟨Boffi–Durán–G.–Gastaldi, '17⟩

$V = H^1_0(\Omega)$, $H = L^2(\Omega)$

$$a(u,v) = \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, dx$$

We discussed a posteriori error indicators for the approximation of elliptic eigenvalue problems with *nonconforming* finite elements

V_h nonconforming piecewise linear element (continuity at the barycenter of faces)

$$a_h(u,v) = \sum_T \int_T \operatorname{grad} u \cdot \operatorname{grad} v \, dx$$

$$\lambda_h \in \mathbb{R},\ u_h \in V_h \qquad a_h(u_h, v) = \lambda_h (u_h, v)_H \quad \forall v \in V_h$$

A posteriori error indicator

Let $\tilde{u}_{i,h}$ be a *conforming* P1 finite element obtained from $u_{i,h}$ by local averaging, then our indicators read

$$\mu_{i,T}^2 = \|\nabla \tilde{u}_{i,h} - \nabla_h u_{i,h}\|_{L^2(T)}^2 \qquad \mu_i^2 = \sum_T \mu_{i,T}^2$$

$$\eta_{i,T}^2 = h_T^2 \|\lambda_{i,h} u_{i,h}\|_{L^2(T)}^2 \qquad \eta_i^2 = \sum_T \eta_{i,T}^2$$

These indicators provide reliability (upper bounds) and efficiency (local lower bounds) for the error in the eigenvalues and in the eigenfunctions

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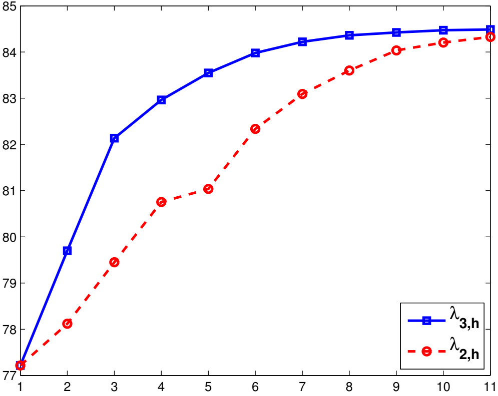
For the square ring problem we are given (at least) three options:

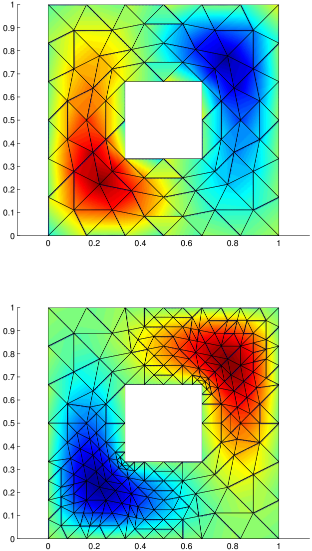
- Indicator based on $(\lambda_{2,h}, u_{2,h})$
- Indicator based on $(\lambda_{3,h}, u_{3,h})$
- Indicator based on both $(\lambda_{2,h}, u_{2,h})$ and $(\lambda_{3,h}, u_{3,h})$

Refinement based on $\lambda_{3,h}$

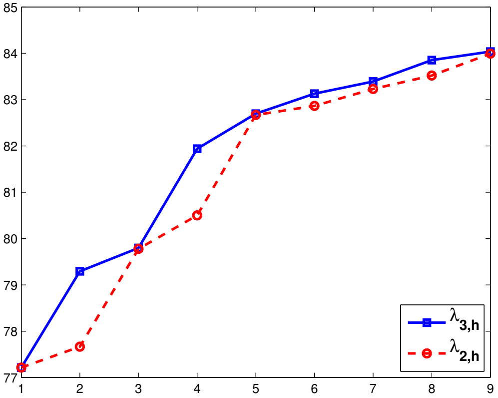
Remark: nonconforming discretization provides eigenvalue approximation from below



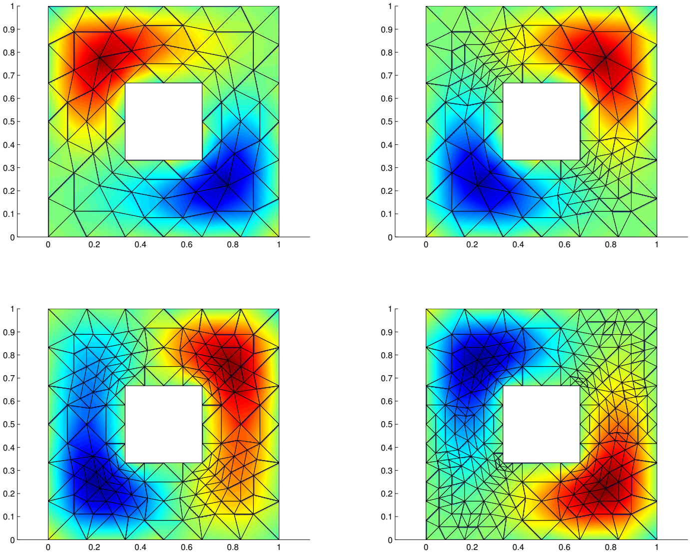
Refinement based on $\lambda_{3,h}$ (eigenfunction $u_{3,h}$)



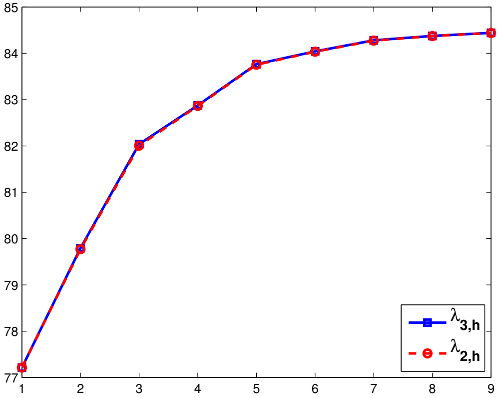
Refinement based on $\lambda_{2,h}$



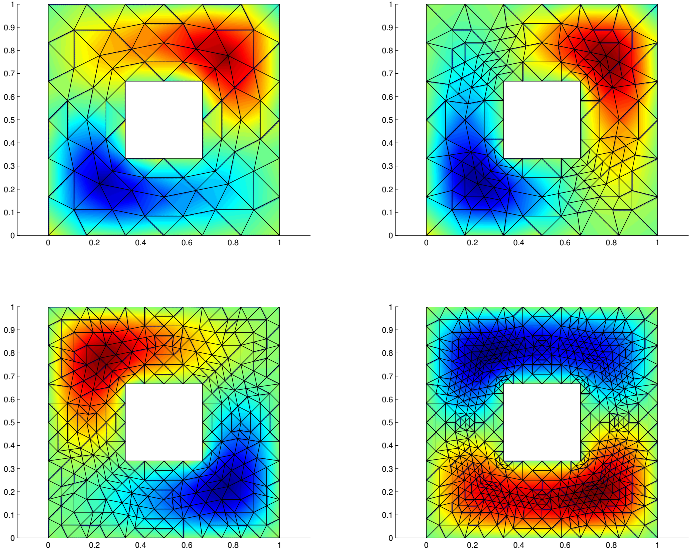
Refinement based on $\lambda_{2,h}$ (eigenfunction $u_{2,h}$)



Refinement based on $\lambda_{2,h}$ and $\lambda_{3,h}$ (eigenvalues)



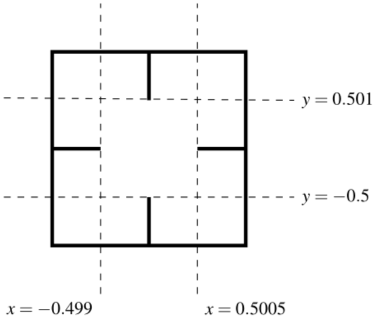
Refinement based on $\lambda_{2,h}$ and $\lambda_{3,h}$ (eigenfunction $u_{2,h}$)

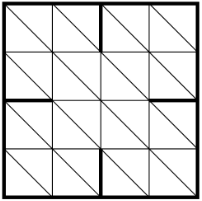


A step forward: cluster of eigenvalues

⟨Gallistl, '14⟩

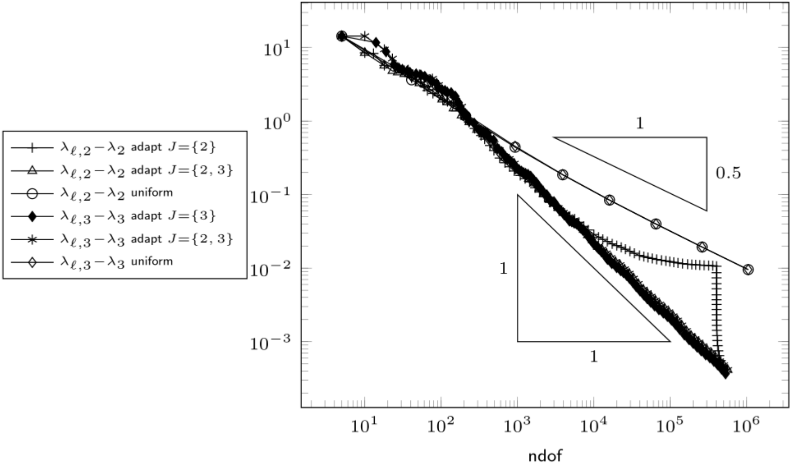
A slightly non-symmetric domain





Now $\lambda_2 < \lambda_3$ but they are very close to each other

Non-symmetric slitted domain



Adaptive FEM for elliptic eigenproblems

Some results (simple eigensolutions)

- ⟨Dai–He–Xu, '08⟩
- ⟨Garau–Morin–Zuppa, '09⟩
- ⟨Giani–Graham, '09⟩
- ⟨Garau–Morin, '11⟩
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Cluster-robust estimates

- Conforming FEM ⟨Gallistl, '14⟩
- Nonconforming FEM ⟨Carstensen–Gallistl–Schedensack, '14⟩
- Morley element for the biharmonic operator ⟨Gallistl ,'14⟩

Outline

Adaptive FEM for eigenvalue problems in mixed form

⟨Boffi–Gallistl–G.–Gastaldi, '17⟩

Find $\lambda \in \mathbb{R}$ and $u \in L^2(\Omega)$ with $u \neq 0$ such that for $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$

$$\begin{cases} \displaystyle\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, d\mathbf{x} + \int_\Omega u \,\mathrm{div}\, \boldsymbol{\tau}\, d\mathbf{x} = 0 & \forall \boldsymbol{\tau} \in H(\mathrm{div};\Omega) \\ \displaystyle\int_\Omega v \,\mathrm{div}\, \boldsymbol{\sigma}\, d\mathbf{x} = -\lambda \int_\Omega uv\, d\mathbf{x} & \forall v \in L^2(\Omega) \end{cases}$$

$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, d\mathbf{x}$ $\qquad$ $b(\boldsymbol{\tau}, v) = \int_\Omega v \,\mathrm{div}\, \boldsymbol{\tau}\, d\mathbf{x}$

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Finite element approximation

$\Sigma_h \subset H(\operatorname{div};\Omega)$ and $M_h \subset L^2(\Omega)$
Find $\lambda_h \in \mathbb{R}$ and $u_h \in M_h$ with $u_h \neq 0$ such that for $\boldsymbol{\sigma}_h \in \Sigma_h$

$$\begin{cases} \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \, d\mathbf{x} + \int_\Omega u_h \operatorname{div} \boldsymbol{\tau} \, d\mathbf{x} = 0 & \forall \boldsymbol{\tau} \in \Sigma_h \\ \int_\Omega v \operatorname{div} \boldsymbol{\sigma}_h \, d\mathbf{x} = -\lambda_h \int_\Omega u_h v \, d\mathbf{x} & \forall v \in M_h \end{cases}$$

A posteriori analysis for the mixed Laplacian

Let $\lambda_{h,j} \in \mathbb{R}$, $\sigma_{h,j} \in \Sigma_h$, $u_{h,j} \in M_h$ denote an eigensolution.

We consider the following error indicator

$$\eta_{h,j}(T)^2 = \|h_T(\sigma_{h,j} - \nabla u_{h,j})\|_T^2 \qquad \text{1}^{\text{st}} \text{ equation “}\sigma = \nabla u\text{”}$$

$$\left.\begin{aligned} &+ \|h_T \operatorname{curl} \sigma_{h,j}\|_T^2 \\ &+ \sum_{E \in \mathcal{E}(T)} h_E \|[\sigma_{h,j}]_E \cdot t_E\|_E^2 \end{aligned}\right\} \text{2}^{\text{nd}} \text{ residual term}$$

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2nd residual term

Take $\tau = \operatorname{curl} \varphi$

$$(\sigma - \sigma_h, \tau) = -(\sigma_h, \operatorname{curl} \varphi) = -\sum_T \left\{ (\operatorname{curl} \sigma_h, \varphi)_T + \int_{\partial T} (\sigma_h \cdot t) \varphi \, ds \right\}$$

AFEM for clusters of eigenvalues

Cluster of length N

$\lambda_{n+1}, \ldots, \lambda_{n+\mathsf{N}}$

$J = \{n+1, \ldots, n+\mathsf{N}\}$

Corresponding combination of eigenspaces

$W = \mathrm{span}\{u_j \mid j \in J\}$

$W_{\mathcal{T}_h} = W_h = \mathrm{span}\{u_{h,j} \mid j \in J\}$

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How to implement the AFEM scheme

Consider contribution of all elements in W_ℓ simultaneously

$$\theta \sum_{j\in J} \eta_{\ell,j}(\mathcal{T}_\ell)^2 \le \sum_{j\in J} \eta_{\ell,j}(\mathcal{M}_\ell)^2$$

Remark: notation $\mathcal{T}_\ell$ or $\mathcal{T}_h$, $\mathcal{T}_H$

Error quantity

Let us introduce the gradient $\mathbf{G}$ and the discrete gradient $\mathbf{G}_h$

$\mathbf{G}(w) \in H(\mathrm{div};\Omega)$ is the solution to

$$a(\mathbf{G}(w),\tau) + b(\tau, w) = 0 \quad \text{for all } \tau \in H(\mathrm{div};\Omega)$$

$\mathbf{G}_h(w_h) \in \Sigma_h$ is the solution to

$$a(\mathbf{G}_h(w_h),\tau_h) + b(\tau_h, w_h) = 0 \quad \text{for all } \tau_h \in \Sigma_h.$$

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Error quantity

$$d(v,w) = \sqrt{\|v-w\|^2 + \|\mathbf{G}(v) - \mathbf{G}(w)\|^2}$$

N.B: when v (resp. w) belongs to M_h, then $\mathbf{G}_h(v)$ (resp. $\mathbf{G}_h(w)$) should be used

$$\boxed{\delta(W, W_h) = \sup_{\substack{u\in W\\ \|u\|=1}} \inf_{v_h\in W_h} d(u, v_h)}$$

Theoretical error indicator

Seminorm

$$
\begin{aligned}
|g_h|^2_{\eta,T} = \|h_T(\mathbf{G}_h(g_h) - \nabla g_h)\|^2_T \\
+ \|h_T \operatorname{curl} \mathbf{G}_h(g_h)\|^2_T \\
+ \sum_{E \in \mathcal{E}(T)} h_E \|[\mathbf{G}_h(g_h)]_E \cdot t_E\|^2_E,
\end{aligned}
$$

so that

$$\eta_{h,j}(T) = |u_{h,j}|_{\eta,T}.$$

Let (λ, σ, u) be an eigensolution to the continuous problem, then

$$\boxed{\mu_h(u; T) = |\Lambda_h u|_{\eta,T}}$$

where $\Lambda_h = P_h^W \circ T_h^\lambda$ (see next page)

Useful operators

P_h^W is the L^2-projection onto W_h

$T_h^\lambda : L^2 \to M_h$ is defined by

$$\begin{cases} a(\mathbf{G}_h(T_h^\lambda g), \tau_h) + b(\tau_h, T_h^\lambda g) = 0 & \forall \tau_h \in \Sigma_h \\ b(\mathbf{G}_h(T_h^\lambda g), v_h) = -(\lambda g, v_h) & \forall v_h \in M_h \end{cases}$$

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Then, we have

$$\Lambda_h = P_h^W \circ T_h^\lambda = T_h^\lambda \circ P_h^W$$

that is,

$$\begin{cases} a(\mathbf{G}_h(\Lambda_h u), \tau_h) + b(\tau_h, \Lambda_h u) = 0 & \forall \tau_h \in \Sigma_h \\ b(\mathbf{G}_h(\Lambda_h u), v_h) = -(\lambda P_h^W u, v_h) & \forall v_h \in M_h \end{cases}$$

Main theorem (convergence and optimal rate)

Nonlinear approximation classes ⟨Binev–Dahmen–DeVore, '04⟩
⟨Cascon–Kreuzer–Nochetto–Siebert, '08⟩

Best convergence rate $s \in (0, +\infty)$ characterized in terms of

$$|W|_{\mathcal{A}_s} = \sup_{m \in \mathbb{N}} m^s \inf_{\mathcal{T} \in \mathbb{T}(m)} \delta(W, W_{\mathcal{T}}).$$

In particular, $|W|_{\mathcal{A}_s} < \infty$ if $\delta(W, W_{\mathcal{T}}) = O(m^{-s})$ for the optimal triangulations in $\mathbb{T}(m)$, that is, with $\mathrm{card}(\mathcal{T}) - \mathrm{card}(\mathcal{T}_0) \leq m$

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Theorem (Boffi–Gallistil–G.–Gastaldi, '17)

Provided the initial mesh-size and the bulk parameter θ are small enough, if for the eigenvalue cluster W it holds $|W|_{\mathcal{A}_s} < \infty$, then the sequence of discrete clusters W_ℓ computed on the mesh $\mathcal{T}_\ell$ satisfies the optimal estimate

$$\delta(W, W_\ell)(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^s \leq C|W|_{\mathcal{A}_s}$$

Convergence of the eigenvalues

The previous theorem implies that the eigenfunctions in the cluster are optimally approximated. The next theorem shows that the eigenvalues are well approximated as well

Theorem (Boffi–Gallistil–G.–Gastaldi, '17)

Let J denote the set of indices corresponding to the eigenvalues in the cluster W. Then

$$\sup_{i\in J}\inf_{j\in J}|\lambda_i-\lambda_{\ell,j}|\leq C\delta(W,W_\ell)^2$$

Convergence of the eigenvalues (sketch of the proof)

T and T_ℓ solution operators

$E : L^2 \to L^2$ orthogonal projection onto W
$E_\ell : L^2 \to L^2$ orthogonal projection onto W_ℓ

$F_\ell = E_\ell|_W$

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For ℓ large enough F_ℓ is a bijection from W to W_ℓ

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Some useful estimates

$$\|(T - T_\ell)x\| \leq C\delta(W, W_\ell)$$
$$\|(A - A_\ell)x\|_{\mathrm{div}} \leq C\delta(W, W_\ell)$$

Convergence of the eigenvalues (cont'ed)

Define the following operators

$$\hat{T} = T|_W, \quad \hat{T}_\ell = F_\ell^{-1} T_\ell F_\ell$$

so that the eigenvalues of $\hat{T}$ ($\hat{T}_\ell$, resp.) are equal to $\mu_j = 1/\lambda_j$ ($\mu_{\ell,j} = 1/\lambda_{\ell,j}$ resp.), $j \in J$

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The following estimate holds true

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The operators $\hat{T}$ and $\hat{T}_\ell$ can be represented by symmetric positive definite matrices of dimension $\mathrm{N} \times \mathrm{N}$ (N being the dimension of W)
Standard matrix perturbation theory gives the final result

Convergence of the eigenvalues (cont'ed)

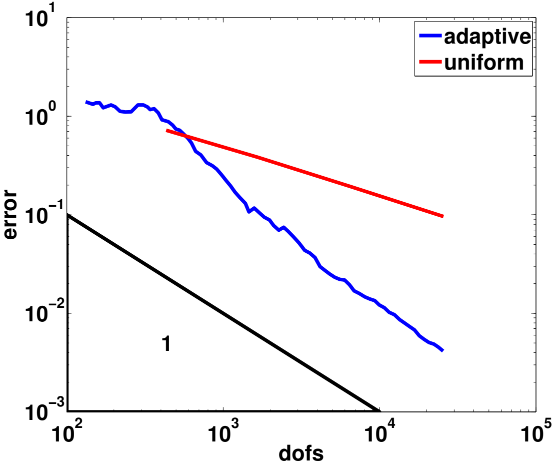
Theorem

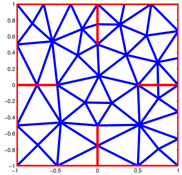
Let J denote the set of indices corresponding to the eigenvalues in the cluster W. Then

$$\sup_{i\in J}\inf_{j\in J}|\lambda_i-\lambda_{\ell,j}|\le C\delta(W,W_\ell)^2$$

Numerical results (non-symmetric slitted domain)

Convergence plot for the second eigenfunction (indicator based on both eigenvalues in the cluster)





THANK YOU

Superconvergence

Let Π_h denote the orthogonal projection onto M_h

Proposition (Superconvergence for the eigenvalue problem)

Any eigensolution (λ, σ, u) *in the cluster satisfies*

$$\|\Pi_h u - \Lambda_h u\| \leq \rho(h) \|\sigma - \mathbf{G}_h(T_h^\lambda u)\|_{\text{div}}$$

with $\rho(h)$ *tending to zero as* h *goes to zero*

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Lemma (Bound for the $\boldsymbol{H}(\mathrm{div})$ norm)

Any eigensolution $(\lambda, \sigma, u) \in \mathbb{R} \times \boldsymbol{\Sigma} \times M$ *satisfies*

$$\|\sigma - \mathbf{G}_h(\Lambda_h u)\|_{\mathrm{div}} \lesssim \|\sigma - \mathbf{G}_h(\Lambda_h u)\| + (1 + \lambda)\|u - \Lambda_h u\|.$$

Efficiency and reliability

Proposition (Efficiency)

$$\mu_h(u;\mathcal{T}_h) \leq C_{\mathrm{eff}}\, d(u, \Lambda_h u)$$

Proposition (Discrete reliability)

Provided the mesh-size of $\mathcal{T}_H$ is sufficiently small, we have

$$\begin{aligned}
&\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| + \|\Lambda_h u - \Lambda_H u\| \\
&\qquad \leq C_{\mathsf{drel}}\mu_H(u;\mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u,\Lambda_h u) + d(u,\Lambda_H u))
\end{aligned}$$

Corollary (Reliability)

Provided the initial mesh-size is sufficiently fine, we have

$$\sum_{j\in J} d(u_j, \Lambda_h u_j)^2 \leq C_{\mathsf{rel}}^2 \sum_{j\in J} \mu_h(u_j, \mathcal{T}_h)^2$$

Quasi-orthogonality

Proposition (Quasi-orthogonality)

There exists a constant C_{qo} *such that*

$$\begin{aligned} d(\Lambda_h u, \Lambda_H u)^2 \leq\ & d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2 \\ & + C_{\mathrm{qo}}\, \rho(h)\big(d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2\big) \end{aligned}$$

Equivalence of estimators and contraction property

N eigenvalues contained in $[A, B]$

Lemma (Local comparison of the error estimators)

Provided the initial mesh-size is small enough, for any $T \in \mathcal{T}_h$

$$N^{-1} \sum_{j \in J} \mu_h(u_j; T)^2 \leq \left(\frac{B}{A}\right)^2 \sum_{j \in J} \eta_{h,j}(T)^2 \leq \left(\frac{B}{A}\right)^2 (2N+4N^2) \sum_{j \in J} \mu_h(u_j; T)^2$$

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Proposition (Contraction property)

Set $\xi_\ell^2 = \sum_{j\in J} \mu_\ell(u_j, \mathcal{T}_\ell)^2 + \beta \sum_{j\in J} d(u_j, \Lambda_\ell u_j)^2$

Provided the initial mesh-size is sufficiently small, there exist $\rho_2 \in (0, 1)$ *and* $\beta \in (0, +\infty)$ *such that*

$$\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2 \quad \textit{for all } \ell \in \mathbb{N}$$