Spectral geometry - from the 19th to 21st century in 50 minutes

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Spectral Geometry: Theory, Numerical Analysis and Applications (18w5090)

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Main question of Spectral Geometry:

What are the relations between the eigenvalues and the geometry of Ω?

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I will address some of particular questions and some answers (or their absence!). The choice of topics is NOT a reflection of their importance but rather of my experience!

Lord Rayleigh's *Theory of Sound*, 1877

For the square $S = [0, \pi]^2 \subset \mathbb{R}^2$ the eigenfunctions are

$$u_{j,k}(x, y) = \sin(jx)\sin(ky), \qquad v_{l,m}(x, y) = \cos(\ell x)\cos(my),$$

with eigenvalues

$$\begin{cases} \lambda_{j,k}(S) = j^2 + k^2, \\ \quad j, k \in \mathbb{N}, \end{cases} \qquad \begin{cases} \mu_{\ell,m}(S) = \ell^2 + m^2, \\ \quad \ell, m \in \mathbb{N} \cup \{0\}. \end{cases}$$

Variables also separate for the unit disk $B_d = B_d(1) =\subset \mathbb{R}^d$, with eigenfunctions ($d = 2$):

$$u_{j,k}(r, \phi) = J_{|j|}(j_{j,k} r)\mathrm{e}^{\mathrm{i}j\phi}, \qquad v_{\ell,m}(r, \phi) = J_{|\ell|}(j'_{\ell,m} r)\mathrm{e}^{\mathrm{i}\ell\phi},$$

and eigenvalues

$$\begin{cases} \lambda_{j,k}(B_2) = j_{j,k}^2, \\ j \in \mathbb{Z}, k \in \mathbb{N}, \end{cases} \qquad \begin{cases} \mu_{\ell,m}(B_2) = {j'}_{\ell,m}^2, \\ \ell \in \mathbb{Z}, k \in \mathbb{N} \end{cases}$$

(and additionally $\mu_{0,0} = 0$ with $v_{0,0} = 1$).

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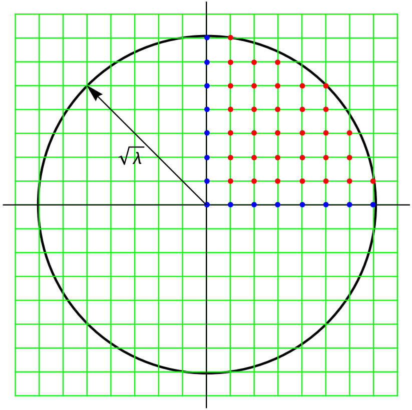
It is in fact easier to look at the inverse function — n as a function of λ, or more precisely at the counting functions

$$N_{\mathrm{Dir}}(\Omega; \lambda) = \#\{n : \lambda_n(\Omega) \le \lambda\}.$$

$$N_{\mathrm{Neu}}(\Omega; \lambda) = \#\{n : \mu_n(\Omega) \le \lambda\}.$$

Counting integer points

For the square S, the eigenvalues less than λ are the squared distances to the origin from the integer lattice points inside the circle of radius $\sqrt{\lambda}$:



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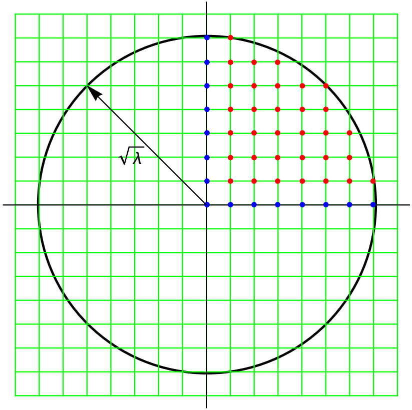

Hence $N_{\mathrm{Dir}}(S;\lambda) \approx N_{\mathrm{Neu}}(S;\lambda) \approx \frac{1}{4}\left|B_2\left(\sqrt{\lambda}\right)\right|_2 = \frac{\pi\lambda}{4}$ as $\lambda \to \infty$,

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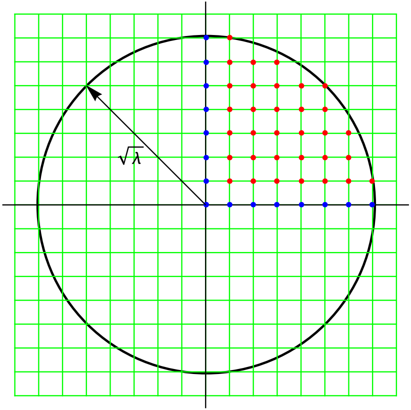

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(Rayleigh made a mistake here, corrected by Sir James Hopwood Jeans).

We can re-write this as $N_{\text{Dir, Neu}}(S;\lambda) \approx \frac{1}{4\pi}|S|_2\lambda$ as $|S|_2 = \pi^2$.

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$$N_{Dir,\ Neu}(\Omega;\lambda) = \underbrace{(2\pi)^{-d}\omega_d|\Omega|_d\lambda^{d/2}}_{W_d(\Omega;\lambda)} + o\left(\lambda^{d/2}\right),\ \omega_d = |B_d|_d.$$

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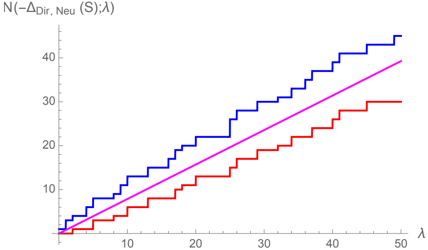

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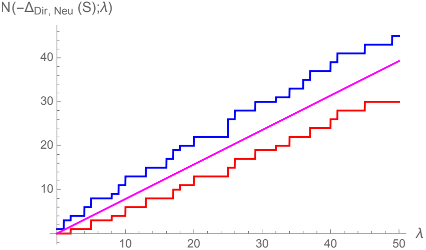

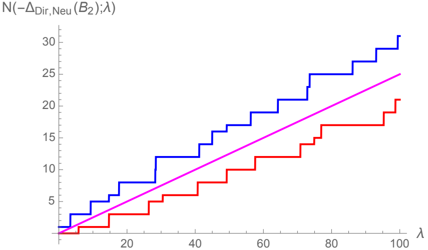

Two-term Weyl's Law

Weyl's Law cannot work very well as it does not take account of the boundary or boundary conditions. So he himself conjectured that in 2d,

$$N_{\text{Dir, Neu}}(\Omega;\lambda) = \frac{1}{4\pi}|\Omega|_2\lambda \mp \frac{1}{4\pi}|\partial\Omega|_1\sqrt{\lambda} + o\left(\sqrt{\lambda}\right),$$

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$$N_{\text{Dir, Neu}}(\Omega;\lambda) = W(\Omega,\lambda) \mp C|\partial\Omega|_{d-1}\lambda^{(d-1)/2} + o\left(\lambda^{(d-1)/2}\right),$$

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Two-term Weyl's asymptotic formula holds subject to the so-called non-periodicity condition on Ω.

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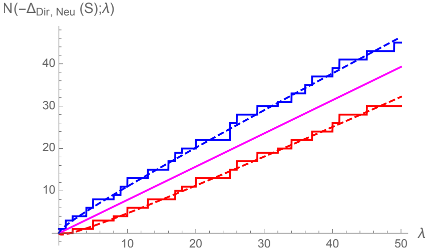

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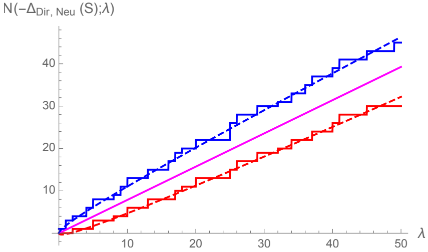

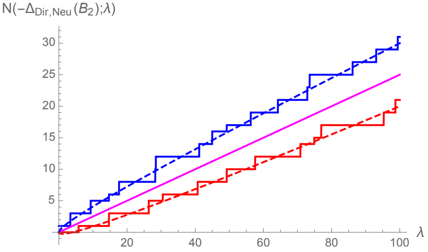

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It is proven for any domain which tiles the space $\mathbb{R}^d$. Some weaker versions of this, e.g.

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Open problem:

Prove Polya's conjecture for disks (and balls).

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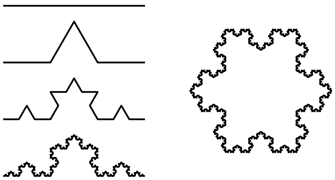

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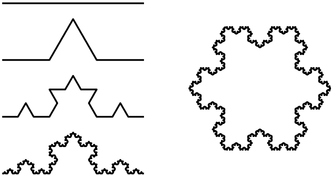

Theorem (Brossard–Carmona; also Lapidus, ML–Vassiliev)

If the boundary $\partial\Omega$ has the (interior) Minkowski dimension $\mathfrak{d} \in [d-1, d)$, then

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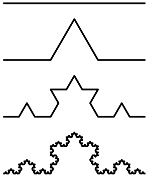

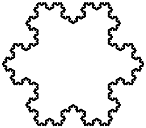

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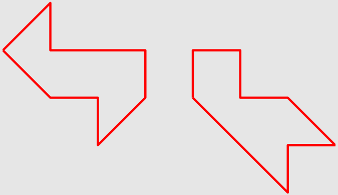
Answer (Gordon–Webb–Wolpert)



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and running ahead

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Construct an example of Steklov-isospectral non-isometric planar domains.

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Prove that for $d = 2$, $\lambda_1(\Omega)$ is minimised among all polygons with p sides and given area by the regular p-gon, for $p \geq 5$.

General minimisers

Theorem (Bucur; Mazzoleni-Pratelli 2011)

The problem $\min\{\lambda_n(\Omega), \Omega \subset \mathbb{R}^d, |\Omega|_d = c\}$ *has a solution* $\Omega^*_{d,n}$ *which is a bounded open set with finite* $|\partial\Omega|_{d-1}$.

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I want to mention one more fact:

Multiplicity of the optimal eigenvalue

$\lambda_n(\Omega^*_{d,n})$ is multiple subject to Schiffer's conjecture.

Schiffer's conjecture and the Pompeiu problem

Schiffer's conjecture

The over-determined spectral problem

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has a non-trivial solution if and only if Ω is a ball.

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The Pompeiu problem

If there exists a non-zero continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and a simply connected domain $\Omega \subset \mathbb{R}^d$ such that $\int_{\Omega'} f = 0$ for all $\Omega' \sim \Omega$, then Ω is a ball.

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Friedlander's inequality

From the variational principles for the Dirichlet and Neumann eigenvalues,

$$\left\{\begin{matrix}\lambda_n\\ \mu_n\end{matrix}\right\} = \inf_{\substack{S\subset\left\{\begin{matrix}H^1_0(\Omega)\\ H^1(\Omega)\end{matrix}\right\}\\ \dim S=n}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{\|\nabla u\|^2}{\|u\|^2},$$

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$$\mu_n(\Omega) \leq \lambda_n(\Omega).$$

But we also have

Theorem (Freidlander, 1991)

For any $\Omega \in \mathbb{R}^d$, and any n,

$$\mu_{n+1}(\Omega) \leq \lambda_n(\Omega).$$

Proofs of Friedlander's inequality

Proof 2: Filonov, 2004.

Consider a test-space

$$S = \left\{u_1, \ldots, u_n, \mathrm{e}^{\mathrm{i}\tau\cdot\mathbf{x}}\right\}, \quad |\tau| = \sqrt{\lambda_n}$$

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Proof 1: Friedlander

Relies on the so-called *Dirichlet-to-Neumann map* $\mathcal{D}_\lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, which sends f into $\left.\dfrac{\partial U}{\partial n}\right|_{\partial\Omega}$ where U solves

$$-\Delta U = \lambda u, \qquad U|_{\partial\Omega} = f.$$

$\mathcal{D}_\lambda$ is defined for $\lambda \notin \mathrm{Spec}(-\Delta_{\mathrm{Dir}})$. $\mathcal{D}_\lambda$ is an elliptic self-adjoint operator, and for a smooth boundary $\partial\Omega$ is a pseudodifferential operator of order one with principal symbol $|\xi|$. $\mathcal{D}_0$ is called the *Steklov operator*.

Eigenvalues of DN map

Let $\sigma_1(\lambda) \le \sigma_2(\lambda) \le \dots$ denote the eigenvalues of the DN map, i.e. those σ for which there is a non-trivial solution of

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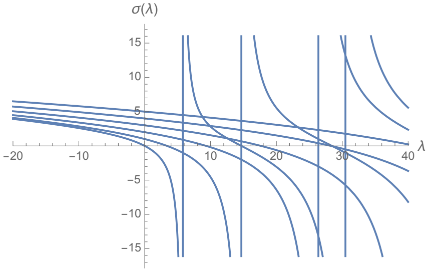

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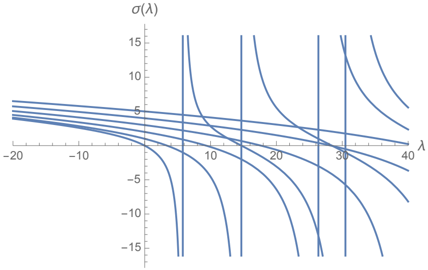

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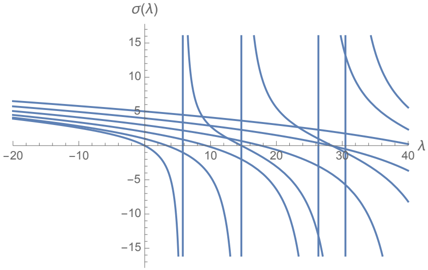

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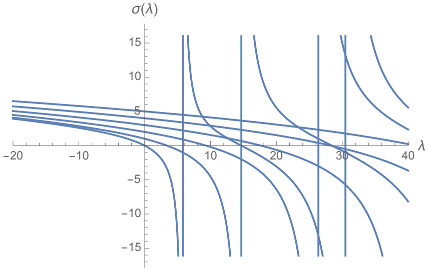

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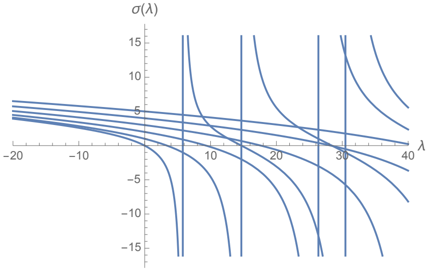

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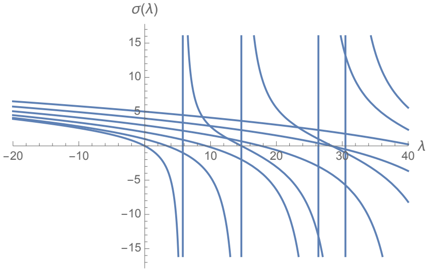

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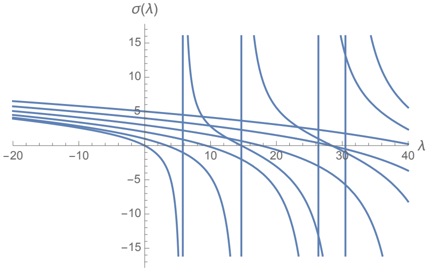

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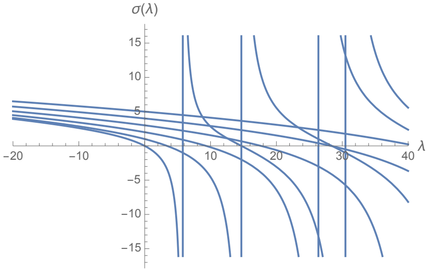

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Therefore $\#\{\sigma(\lambda) < 0\} = N_{\mathrm{Neu}}(\lambda) - N_{\mathrm{Dir}}(\lambda)$. Friedlander then proved that $\#\{\sigma(\lambda) < 0\} \ge 1$ for $\lambda > 0$.

Weinberger's conjecture

Remark

Friedlander's bound does not generally hold on a Riemannian manifold.

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Some time ago we tried to improve Friedlander's bound via Filonov's approach by adding extra exponentials to the trial set. But then one needs to kill cross products of exponentials. So an interesting object arises:

Null variety

$$\mathcal{N}(\Omega) := \left\{\xi \in \mathbb{C}^d : \int_\Omega \mathrm{e}^{\mathrm{i}\xi\cdot\mathbf{x}} = 0\right\}$$

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With Benguria and Parnovski, we studied

$$\kappa(\Omega) := \mathrm{dist}(\mathcal{N}(\Omega), \mathbf{0}) > 0.$$

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Strange things come together!

$\mathcal{N}(\Omega)$ is important! — Schiffer's conjecture would fail for Ω such that $\mathcal{N}(\Omega)$ contains a (large) sphere.

More on DN maps

$\mathcal{D}_\lambda$ and in particular $\mathcal{D}_0$ are fascinating objects with many interesting properties and important applications. One particular application: numerical domain decomposition for non-compact domains or manifolds (with regular ends).

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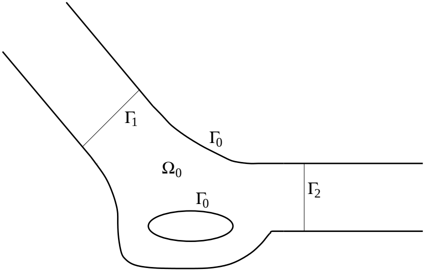

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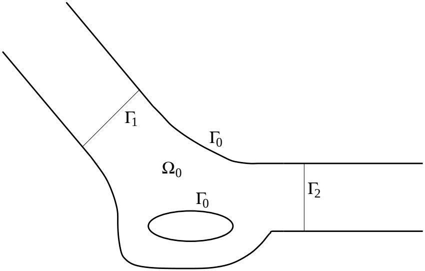

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Ω is decomposed into a compact part Ω_0 and non-compact "ends" $\Omega_{\text{ext}} = \Omega_1 \cup \Omega_2 (\cup \dots)$, separated by the "interface" $\Gamma = \Gamma_1 \cup \Gamma_2 (\cup \dots)$.

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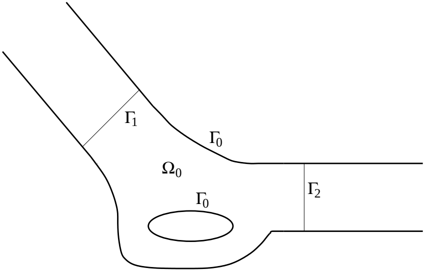

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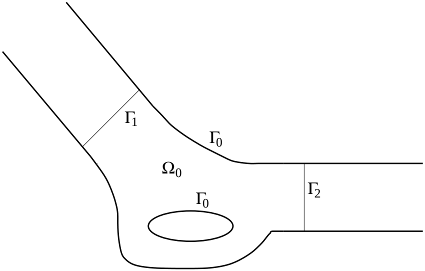

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Typical spectral picture: continuous spectrum $[\lambda_0, +\infty)$ ("bounded solutions"), maybe some eigenvalues either below or embedded into the continuous spectrum ("decreasing solutions"), and complex resonances ("growing solutions").

Numerical scheme for finding eigenvalues and resonances

- Re-write the problem as $\mathcal{D}_\lambda^{\text{int}} f = -\mathcal{D}_\lambda^{\text{ext}} f$, where $\mathcal{D}_\lambda^{\text{int; ext}}$ are partial DN maps on the interface Γ, with appropriate conditions at infinity for the exterior one, and $f = u|_\Gamma$.
- More precisely, consider the pencil

$$\mathcal{A}_\lambda(t) = (\mathcal{D}_\lambda^{\text{int}})^{-1} + t(\mathcal{D}_\lambda^{\text{ext}})^{-1},$$

  acting on functions on Γ. Then λ is a candidate to be an eigenvalue/resonance if $t = 1$ is an eigenvalue of the pencil $\mathcal{A}_\lambda$.
- For embedded eigenvalues, check additionally the orthogonality conditions to the "eigenfunctions" of the continuous spectrum.
- Use monotonicity of $\mathcal{D}_\lambda$ in λ.

Representation formula for $(\mathcal{D}_\lambda^{\text{int}})^{-1}$

Easy way to compute the partial Neumann-to-Dirichlet map.

- Choose a basis $\{w_j\}$ in $L^2(\Gamma)$.
- Then

$$\langle (\mathcal{D}_\lambda^{\text{int}})^{-1} w_j, w_k \rangle_\Gamma = \sum_{i=1}^{\infty} \frac{\langle w_j, v_i \rangle_\Gamma \langle v_i, w_k \rangle_\Gamma}{\mu_i - \lambda},$$

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or formally

$$(\mathcal{D}_\lambda^{\mathrm{int}})^{-1} = S(M - \lambda I)^{-1} S^*,$$

$$M = \mathrm{diag}(\mu_1, \mu_2, \dots), \quad S = (\langle w_j, v_i \rangle_\Gamma)_{j,i=1}^{\infty}.$$

Existence of trapped modes

What are the main questions theoretically?

- What are sufficient conditions on the geometry of Ω for the existence of embedded eigenvalues?
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Theorem (Evans–ML–Vassiliev, 1994)

Let Ω be a strip $(\mathbb{R} \times (-1, 1)) \setminus \mathcal{O}$, $\mathcal{O}$ is a compact obstacle. If $\mathcal{O}$ is symmetric with respect to the central line $y = 0$, then $-\Delta_{Neu}(\Omega)$ has an eigenvalue embedded into the continuous spectrum $[0, +\infty)$.

Many more examples, essentially the industry now.

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Open problem:

Create some other examples of the global absence of eigenvalues!

Example: Hyperbolic surfaces of genus one with one cusp

The space of all surfaces of of constant negative curvature -1 and one cusp is two dimensional and can be parameterised by the two Fenchel-Nielsen coordinates $\ell > 0$ and $\tau \in [0, 1]$. The parameter ℓ is the length of a primitive closed geodesic and the angle τ is the twist parameter along this geodesic.

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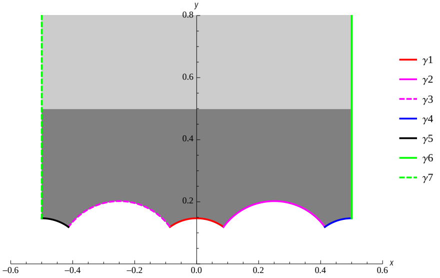

$$-y^2\Delta v = s(1-s)v.$$

joint work in progress with Alex Strohmaier