On the existence and regularity of non-flat profiles for a Bernoulli free boundary problem

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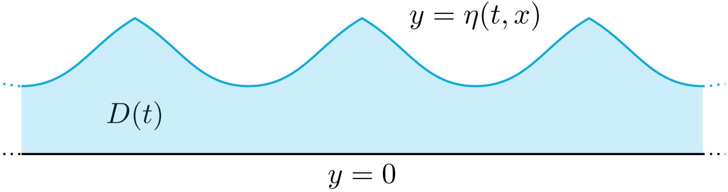
Joint work with Giovanni Leoni

Formulation of the physical problem

Consider a 2-D periodic wave traveling at constant speed c over a flat impermeable bed, in a flow of zero vorticity.



Assume that the fluid is inviscid and incompressible.
Neglect surface tension.
Gravity is the only restoring force.

Passing to a moving frame of reference, the equations of motion can be rewritten as

conservation of momentum	$\begin{cases} \rho((u-c)u_x + vu_y) = -P_x \\ \rho((u-c)v_x + vv_y) = -P_y - \rho g \end{cases}$
conservation of mass	$\nabla \cdot (\rho(u,v)) = 0$
irrotationality	$v_x = u_y$
kinetic boundary conditions	$\begin{cases} v = (u-c)\eta' & \text{on } y = \eta(x) \\ v = 0 & \text{on } y = 0 \end{cases}$
dynamic boundary condition	$P = P_{\mathrm{atm}}$ on $y = \eta(x)$
Bernoulli's equation	$\dfrac{\lvert(u,v)\rvert^2}{2} + gy + \dfrac{P}{\rho} = \mathrm{const.}$ on streamlines.

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Assume that $\rho \equiv 1$, let λ be the length of one wave cycle and set

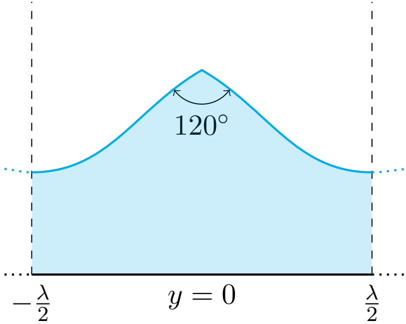
$$\Omega := \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right) \times (0, \infty).$$

Then we can rewrite the system in terms of a stream function ψ:

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \cap \{\psi > 0\}, \\ \psi = 0 & \text{on } \Omega \cap \partial\{\psi > 0\}, \\ |\nabla\psi| = \sqrt{(\text{const.} - 2gy)_+} & \text{on } \Omega \cap \partial\{\psi > 0\}, \\ \psi = m & \text{on } y = 0. \end{cases} \tag{FBP}$$

Stokes conjecture

Stokes, 1847: conjectured the existence of a wave of greatest height, with a has sharp crests of included angle $\frac{2\pi}{3}$.



Why is 120° the expected Stokes angle?

- If v solves

$$\begin{cases} \Delta v = 0 & \text{in } S, \\ v = 0 & \text{on } \partial S, \end{cases}$$

where S is the sector of opening angle ω, then

$$v \sim r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi\theta}{\omega}\right).$$

(see Dauge, Grisvard, Kondratev & Oleinik, ...)

- Bernoulli's condition:

$$r^{\frac{\pi}{\omega}-1} \sim |\nabla v| \sim r^{1/2}.$$

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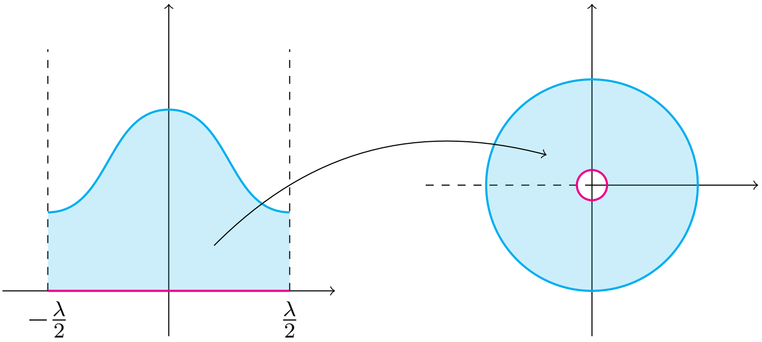
A little bit of history

- Nekrasov '22 mapped the fluid into an annulus by a hodograph transform:



$$\phi(s) = \frac{1}{3\pi}\int_{-\pi}^{\pi} \frac{\sin\phi(t)}{\mu^{-1} + \int_0^t \sin\phi(u)\,du} \log\left|\frac{\operatorname{sn}(\pi^{-1}K(s+t))}{\operatorname{sn}(\pi^{-1}K(s-t))}\right| dt.$$

- Krasovskiĭ '61: for $\mu > \overline{\mu}$ there exists ϕ, a continuous solution of Nekrasov's equation; ϕ has smooth crest and $0 \leq \phi < \pi/6$.
- Keady & Norbury '78: no solutions for $\mu \leq \overline{\mu}$. For $\mu > \overline{\mu}$ there exist continuous solutions with smooth crest and $0 \leq \phi < \pi/2$.
- Toland '78 & McLeod '79: $\{\phi_{\mu_n}\}_n$ converges to a solution of the limiting problem ϕ_0 as $\mu_n \to \infty$ (Stokes wave). Moreover, if $\lim_{s\to 0}\phi_0(s)$ exists then it must be $\frac{\pi}{6}$ (Stokes angle).
- Amick, Fraenkel & Toland '82, Plotnikov '82: $\lim_{s\to 0}\phi_0(s)$ exists.

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A variational approach for water waves

- Solutions (FBP) $\Longleftrightarrow$ critical points of the energy functional

$$J_h(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(h-y)_+\right)\, d\boldsymbol{x}, \quad h > 0,$$

defined for u in the convex set

$$\mathcal{K} := \left\{u \in H^1_{\text{loc}}(\Omega) : u \text{ is } \lambda\text{-periodic in } x \text{ and } u(\cdot,0) \equiv m\right\},$$

see Alt & Caffarelli '81.

$$J(u) := \int_{\Omega} \left(|\nabla u|^2 + \chi_{\{u>0\}} Q^2\right) \, d\boldsymbol{x}.$$

Theorem (Alt & Caffarelli '81)

Assume that

- Ω *is a domain with Lipschitz boundary,*
- $\Gamma \subset \partial\Omega$,
- Q *is Hölder continuous and s.t.*

$$Q(\boldsymbol{x}) \geq Q_{\min} > 0 \tag{1}$$

- u_0 *is nonnegative and s.t.* $J(u_0) < \infty$.

Let u *be a minimizer of* J *over* $\widetilde{\mathcal{K}} := \{H^1_{\mathrm{loc}}(\Omega) : u = u_0 \text{ on } \Gamma\}$*. Then* $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$ *and* $\partial\{u > 0\} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$*, for some* $0 < \alpha < 1$.

- Notice that $\sqrt{(h-y)_+}$ does not satisfy assumption (1), so one cannot expect this regularity for minimizers of J_h.

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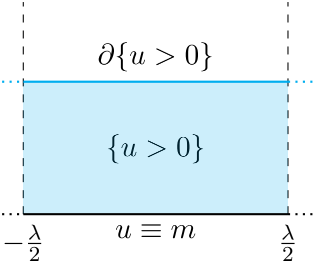
The main drawback of the variational approach

Theorem

Every global minimizer of J_h over $\mathcal{K}$ is a one dimensional function of the form $u = u(y)$.

In particular, only flat profiles can be observed among the free boundaries of global minimizers (see Arama & Leoni '12).



Related works

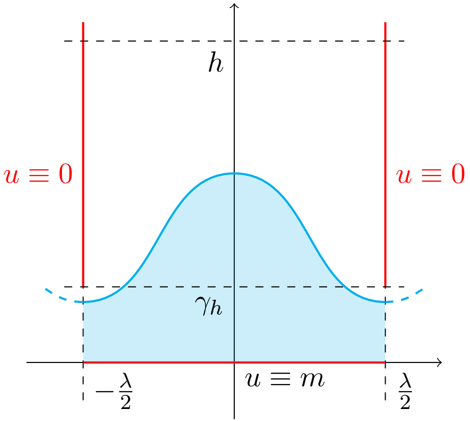
- Arama & Leoni '12: $u \equiv m \rightsquigarrow u = v_0 \in C_c^1((-\lambda/2, \lambda/2))$. This is non-physical. Decay estimates for local minimizers:

$$|\nabla u(\boldsymbol{x})| \leq C r^{1/2}, \quad \boldsymbol{x} \in B_r(\boldsymbol{x}_0),$$

  for $\boldsymbol{x}_0 \in \partial\{u > 0\} \cap \{y = h\}$.
- Varvaruca & Weiss '11, see also Weiss & Zhang '12: If $C = 1 \Rightarrow u$ is a Stokes wave.
- Fonseca, Leoni, Mora '17: Necessary and sufficient minimality conditions in terms of the second variation of J_h for smooth critical points.

Existence of non-flat profiles

By adding an additional Dirichlet boundary condition on part of the later boundary we can construct global minimizers of J_h that are not one dimensional.



We let

$$u_0(x,y) = \frac{m}{\gamma}(\gamma - y)_+$$

and consider the minimization problem for J_h in

$$\mathcal{K}_\gamma := \left\{u \in H^1_{\mathrm{loc}}(\Omega) : u \text{ is } \lambda\text{-periodic in } x \text{ and } u = u_0 \text{ on } \Gamma_\gamma\right\},$$

where $\Gamma_\gamma := \left(\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right] \times \{0\}\right) \cup \left(\left\{\pm\frac{\lambda}{2}\right\} \times (\gamma, \infty)\right)$.

Theorem (G. & Leoni '18: Existence of non-flat minimizers)

Given $m, \lambda, h > 0$, there exists $\overline{\gamma} = \overline{\gamma}(m, \lambda, h) > 0$ such that if $0 < \gamma < \overline{\gamma}$ then every global minimizer $u \in \mathcal{K}_\gamma$ of the functional J_h is not of the form $u = u(y)$.

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Sketch of proof:

Main observation: for flat profiles, if γ is small, the Dirichlet energy plays a predominant role.

- If $w \in \operatorname{argmin}\{J_h(v) : v$ is flat and $\operatorname{supp} v \subset \{y \leq \gamma\}\}$ then

$$J_h(w) \sim \frac{1}{\gamma}$$

- We can construct a competitor u whose energy satisfies

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Existence of a critical height

We now let the parameter h vary and study how this affects the shape of minimizers.

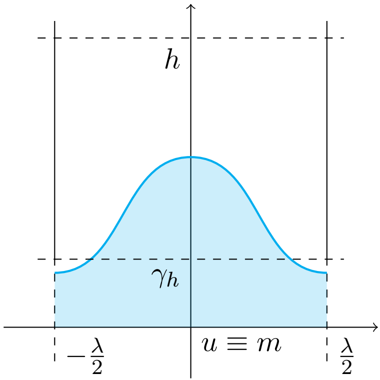
Theorem (G. & Leoni '18: Existence of a critical height)

Let $h \mapsto \gamma_h$ be given as in the previous theorem (i.e. minimizers are not one-dimensional). Then there exists a critical height $0 < h_{\mathrm{cr}} < \infty$ with the property that

(i) *if $h_{\mathrm{cr}} < h < \infty$ then every global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ has support below the line $\{y = h\}$;*

(ii) *if $0 < h < h_{\mathrm{cr}}$ then every global minimizer is positive in $\left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right) \times [h, \infty)$.*

Case $h > h_{\text{cr}}$



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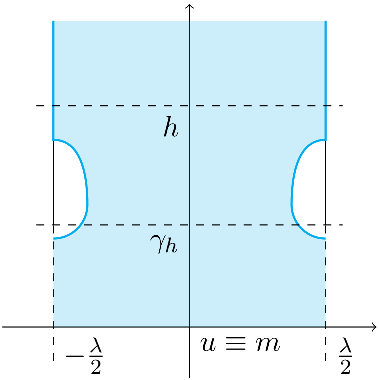

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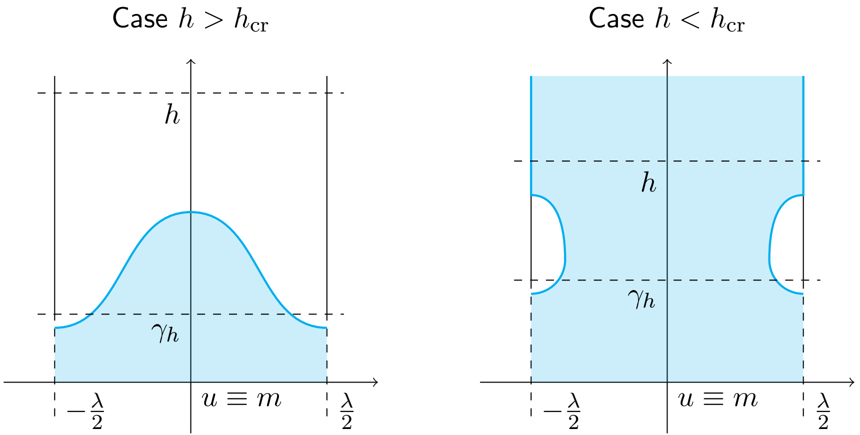
Let u_h, u_δ be global minimizers of J_h and J_δ in $\mathcal{K}_{\gamma_h}$ and $\mathcal{K}_{\gamma_\delta}$, respectively. Then, if $h < \delta$, $\{u_\delta > 0\} \subset \{u_h > 0\}$ and $u_\delta \le u_h$.



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- Stokes waves can only be observed for $h = h_{\text{cr}}$.
- To prove the existence of a Stokes wave we need to show that there is a global minimizer u of $J_{h_{\text{cr}}}$ with support contained in $\{y \le h_{\text{cr}}\}$ and such that $(x, h_{\text{cr}}) \in \partial\{u > 0\}$.
- Idea: Want to find a Stokes wave as the limit of regular waves. (This is reminiscent of the works of Toland and McLeod)

Proposition (Convergence of minimizers)

Let $\{h_n\}_n \subset (0,\infty)$ be s.t. $h_n \nearrow h < \infty$ and for every n let $u_n \in \mathcal{K}_{\gamma_{h_n}}$ be a global minimizer of J_{h_n}. Then there exists a global minimizer u of J_h in $\mathcal{K}_{\gamma_h}$ such that:

- *$u_n \to u$ in $H^1_{\text{loc}}(\Omega)$,*
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Furthermore, u is independ of the sequences $\{h_n\}_n, \{u_n\}_n$.

A similar result holds if $h_n \searrow h > 0$.

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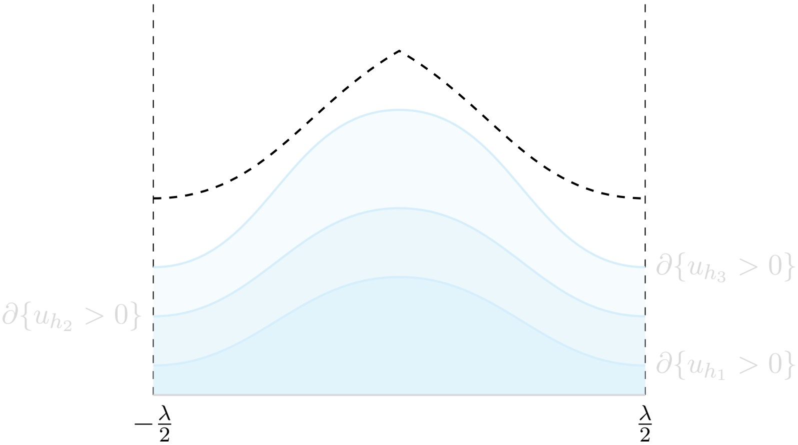

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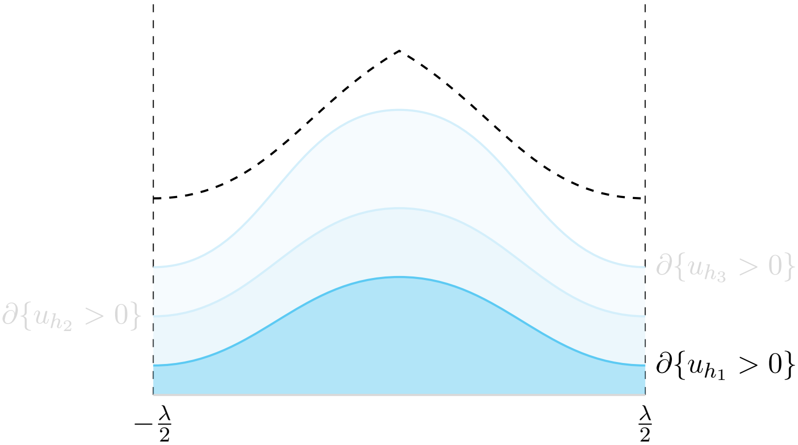

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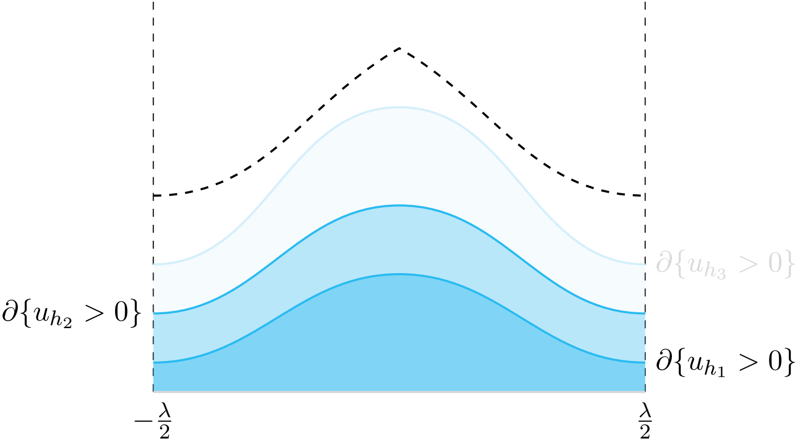

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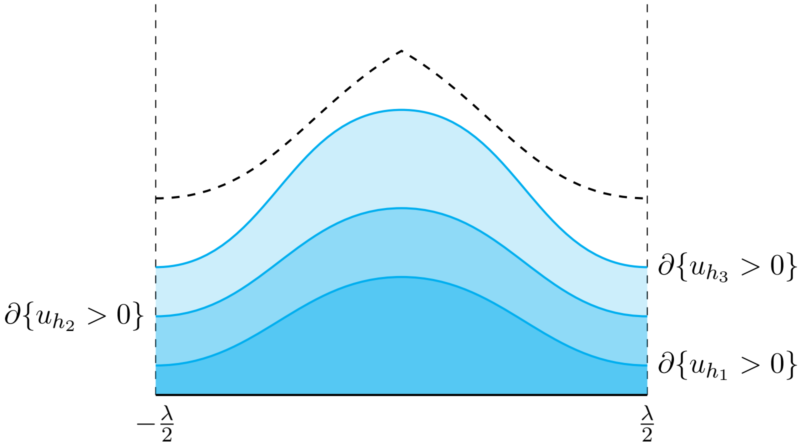

Corollary (Hierarchy of global minimizers)

For every $h > 0$ there are two (possibly equal) global minimizers u_h^+, u_h^- of J_h in $\mathcal{K}_{\gamma_h}$ such that $u_h^- \leq u_h^+$ and if w is another global minimizer then $u_h^- \leq w \leq u_h^+$.

Consider $u_{h_{\rm cr}}^+$ and $u_{h_{\rm cr}}^-$. We can show that:

- the support of $u_{h_{\rm cr}}^-$ is contained in $\{y \leq h_{\rm cr}\}$,
- the support of $u_{h_{\rm cr}}^+$ cannot be strictly below the line $\{y = h_{\rm cr}\}$.
- We have not been able to prove that the support of any global minimizer touches the line $\{y = h_{\rm cr}\}$. This would follow if we had uniqueness at this level.

Theorem (G. & Leoni '18)

There is a unique global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ for all but countably many values of h.

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Scaling of the critical height

- Recall: $u \equiv m$ on $(-\lambda/2, \lambda/2) \times \{0\}$.
- $h_{\text{cr}} \leq \frac{3}{2^{1/3}} m^{2/3}$.
- If m is small enough then

$$h_{\text{cr}} \geq \frac{3k}{2^{2/3}} m^{2/3},$$

where k is the smallest positive root of $27t^3 + 16t - 8 = 0$.

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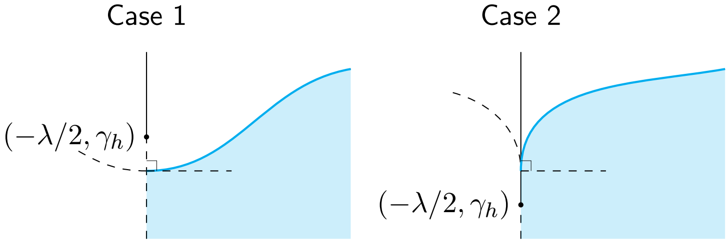
Ongoing work: boundary regularity



- Case 1: regular waves.
- Case 2: Chang-Lara & Savin '17: $\partial\{u > 0\} \in C^{1,1/2}_{\text{loc}}$, non-physical behavior.
- WTS: Case 1 occurs by optimizing γ_h and varying λ.

Future work: variational existence of Stokes waves

- Can we improve the uniqueness result?
- Find optimal γ_h.
- Play with the parameters m, λ.

Thank you for your attention!

Additional references

- Free boundary problems: Alt, Caffarelli & Friedman '84, Caffarelli '87, '88, '89, Caffarelli, Jerison & Kenig '04, Raynor '08, Weiss '99, '04.
- Singularly perturbed problems: Berestycki, Caffarelli, & Nirenberg '90, Caffarelli '95, Danielli & Petrosyan '05, Danielli, Petrosyan, & Shahgholian '03, Gurevich '99, Karakhanyan '06, Karakhanyan '18, Lederman & Wolanski '98, Moreira & Texeira '07.
- "Moving parameters": Alt, Caffarelli & Friedman '82, '83, '85, Fusco & Morini '12.
- Water waves: Constantin & Strauss '04, '10, Constantin, Sattinger & Strauss '06, Constantin, Strauss & Varvaruca '16, Chen, Walsh & Wheeler '16, '18, Kinsey & Wu '18, Plotnikov & Toland '04