Robust Dual Dynamic Programming

(join work with Angelos Tsoukalas and Wolfram Wiesemann)

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DRO Workshop 2018

Inspired by SDDP

Stochastic optimization

- Optimizes expected value
- Requires knowledge of distribution

$$\min_{\boldsymbol{x}} \mathbb{E}_{\mathbb{P}}[f(\boldsymbol{x}, \boldsymbol{\xi})]$$

Robust optimization

- Optimizes for the worst case scenario
- Uses only support information (uncertainty set)

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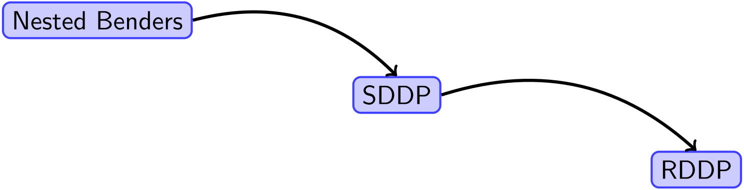

Multistage Robust Optimization

$$\begin{aligned} \text{minimize} \quad & \max_{\boldsymbol{\xi} \in \Xi} \sum_{t=1}^{T} \boldsymbol{q}_t^\top \boldsymbol{x}_t(\boldsymbol{\xi}^t) \\ \text{subject to} \quad & \left. \begin{array}{l} \boldsymbol{T}_t(\boldsymbol{\xi}_t)\, \boldsymbol{x}_{t-1}(\boldsymbol{\xi}^{t-1}) + \boldsymbol{W}_t \boldsymbol{x}_t(\boldsymbol{\xi}^t) \geq \boldsymbol{H}_t \boldsymbol{\xi}_t \\ \boldsymbol{x}_t(\boldsymbol{\xi}^t) \in \mathbb{R}^{n_t},\ \boldsymbol{\xi}^{\boldsymbol{t}} = (\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_t) \end{array} \right\} \forall \boldsymbol{\xi} \in \Xi,\ \forall t \end{aligned}$$

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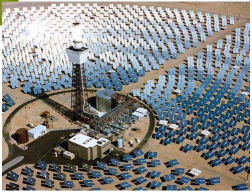
Applications

Hydro Scheduling &
Reservoir Management

Path Planning

Application with
Long Planning Horizons

Long Term Energy
Storage



Nested Formulation

The multistage problem can be expressed through a nested formulation

$$\min_{\boldsymbol{x}_1 \in \mathcal{X}_1} \boldsymbol{q}_1^\top \boldsymbol{x}_1 + \left[\max_{\boldsymbol{\xi}_2 \in \Xi_2} \min_{\boldsymbol{x}_2 \in \mathcal{X}_2(\boldsymbol{x}_1, \boldsymbol{\xi}_2)} \boldsymbol{q}_2^\top \boldsymbol{x}_2 + \left[\cdots + \max_{\boldsymbol{\xi}_T \in \Xi_T} \min_{\boldsymbol{x}_T \in \mathcal{X}_T(\boldsymbol{x}_{T-1}, \boldsymbol{\xi}_T)} \boldsymbol{q}_T^\top \boldsymbol{x}_T \right] \right]$$

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First stage problem

$$\begin{aligned} \min_{\boldsymbol{x}_1 \in \mathbb{R}^{n_1}} \quad & \boldsymbol{q}_1^\top \boldsymbol{x}_1 + \mathcal{Q}_2(\boldsymbol{x}_1) \\ & \boldsymbol{W}_1 \boldsymbol{x}_1 \geq \boldsymbol{h}_1 \end{aligned}$$

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- Polyhedral $\Xi_t \implies$ replace with ext $\Xi_t \implies$ problem decomposes

Approximate Dynamic Programming

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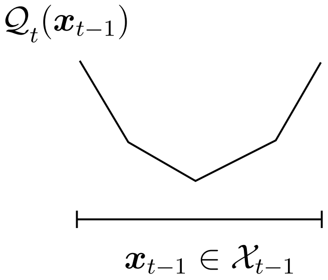

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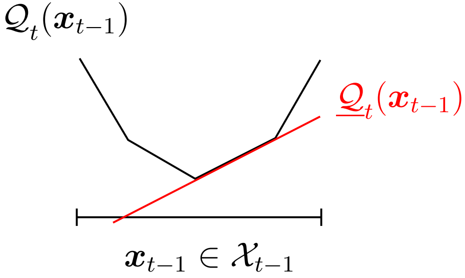

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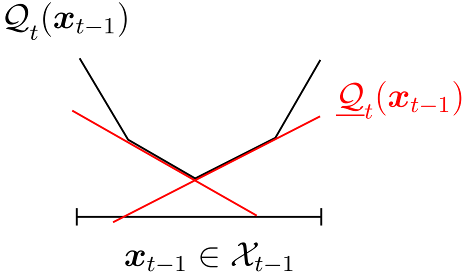
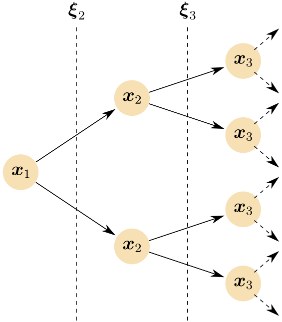


Nested Benders for Robust Optimization



- Maintain outer approximation $\underline{\mathcal{Q}}_t(\boldsymbol{x}_{t-1})$ per node

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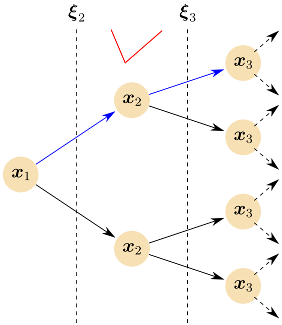

$$x_t^f = \arg \min_{x_t \in \mathbb{R}^{n_t}} \quad q_t^\top x_t + \underline{\mathcal{Q}}_{t+1}(x_t)$$
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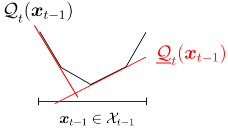

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- **Forward Pass:** Explore one scenario at a time

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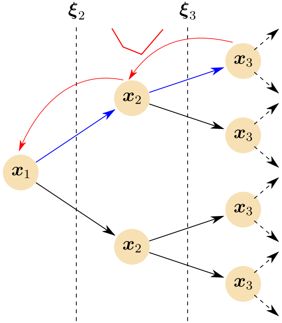

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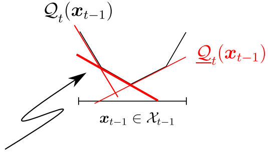

$$[\boldsymbol{H}_t \boldsymbol{\xi}_t - \boldsymbol{T}_t \boldsymbol{x}_{t-1}^f]^\top \boldsymbol{\pi}_t - \underline{\mathcal{Q}}_{t+1}^\star(\boldsymbol{W}_t^\top \boldsymbol{\pi}_t - \boldsymbol{q}_t)$$

- Maintain outer approximation $\underline{\mathcal{Q}}_t(\boldsymbol{x}_{t-1})$ per node
- **Forward Pass:** Explore one scenario at a time
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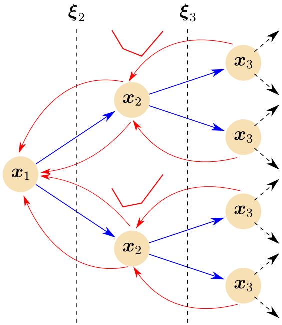

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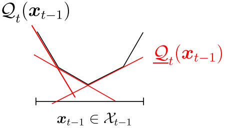

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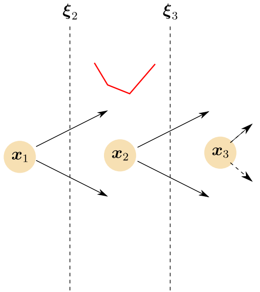
Cut Sharing and SDDP

Exploit the Markov property: Maintain one approximation $\underline{\mathcal{Q}}_t(\boldsymbol{x}_{t-1})$ per stage



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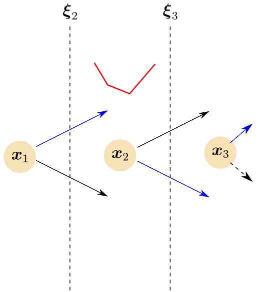

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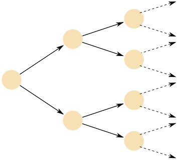

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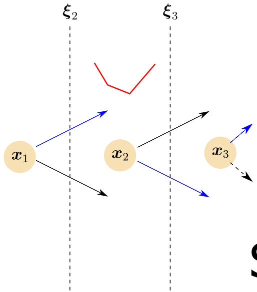

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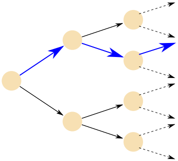

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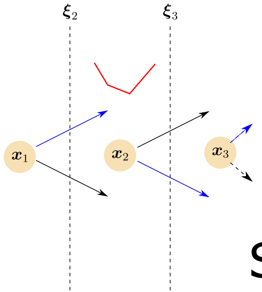

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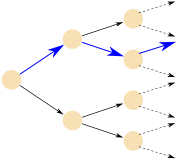
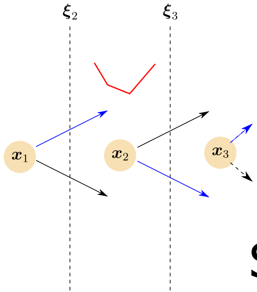


SDDP:

- Small number of refinements
- Good performance in practice

Cut Sharing and SDDP

Exploit the Markov property: Maintain one approximation $\underline{\mathcal{Q}}_t(\boldsymbol{x}_{t-1})$ per stage



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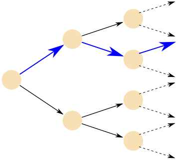

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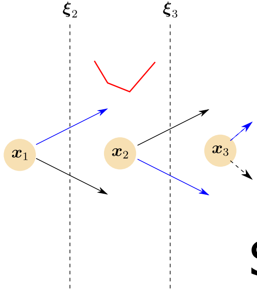

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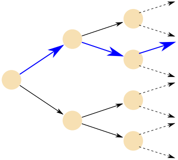

SDDP:

- Small number of refinements
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- Stochastic termination criterion
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- No distributional information for robust optimization

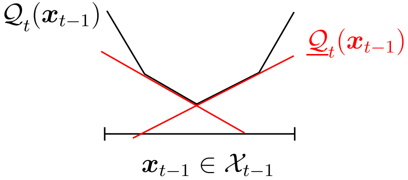
Robust Dual Dynamic Programming (RDDP)

Which scenario/state do we propagate forward?

Robust Dual Dynamic Programming (RDDP)

Main Idea: maintain both

- an outer approximation



Robust Dual Dynamic Programming (RDDP)

Main Idea: maintain both

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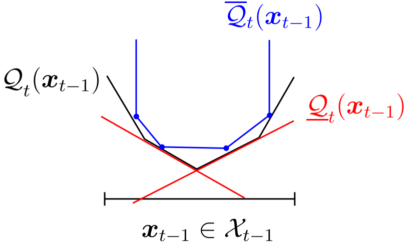
- and an inner approximation



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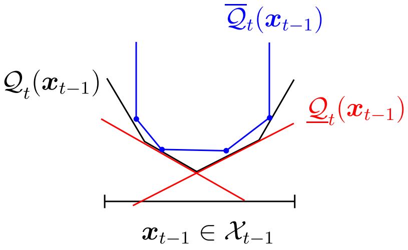

In the forward pass:

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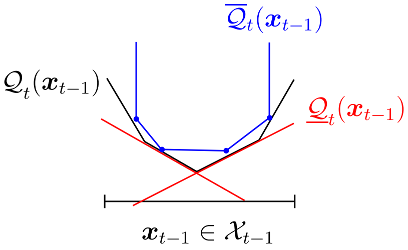

In the forward pass:

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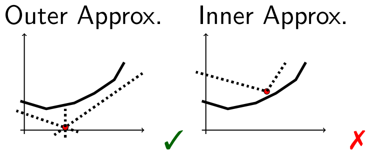
In the backward pass:

- refine both inner and outer approximations

Why Use an Inner Approximation?

Intuitively speaking,

minimizing a convex function



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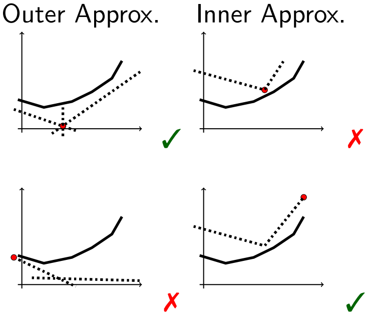

Forward Pass

We want "nature" to be optimistic in its choice, use inner approximation

$$\boldsymbol{\xi}_t^f = \arg \max_{\boldsymbol{\xi}_t \in \text{ext } \Xi_t} \min_{\boldsymbol{x}_t \in \mathbb{R}^{n_t}} \quad \begin{array}{l} \boldsymbol{q}_t^\top \boldsymbol{x}_t + \overline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t) \\ \boldsymbol{T}_t \, \boldsymbol{x}_{t-1}^f + \boldsymbol{W}_t \boldsymbol{x}_t \geq \boldsymbol{H}_t \boldsymbol{\xi}_t \end{array}$$

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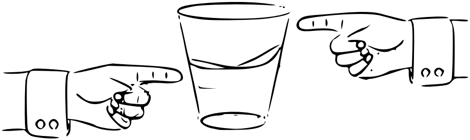
Decision maker



Nature

Backward Pass: Refining Approximation

Inner approximation: Starting with $\boldsymbol{x}_{t-1}^{f}$

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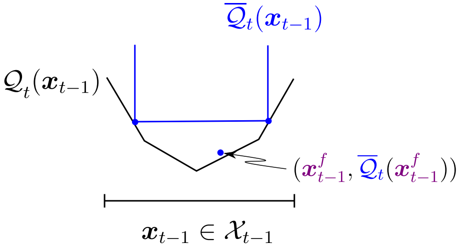

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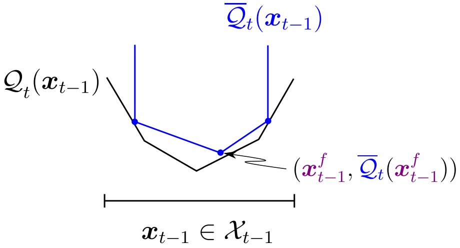

Backward Pass: Refining Approximation

Outer approximation: Starting with $\boldsymbol{x}_{t-1}^{f}$, use $\boldsymbol{\xi}_t^b$ from inner approximation

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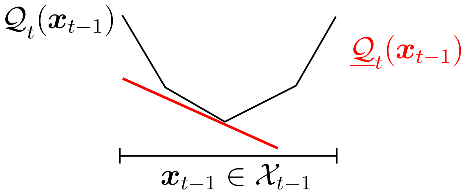

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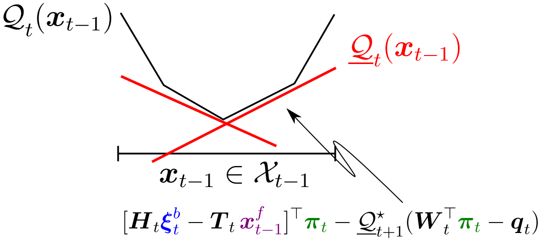

Termination Criterion

Stage-1 Problem:

- using inner approximation get upper bound

$$\overline{J} = \min_{\boldsymbol{x}_1 \in \mathbb{R}^{n_1}} \quad \boldsymbol{q}_1^\top \boldsymbol{x}_1 + \overline{\mathcal{Q}}_2(\boldsymbol{x}_1)$$
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RDDP v.s. Nested Benders & SDDP

Nested Benders:

- Finite convergence
- Deterministic bounds
- No relative complete recourse

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- Implementable strategy at every iteration
- Exponential number of iterations required in worst case

RDDP Extensions

$$
\begin{aligned}
\text{minimize} \quad & \max_{\boldsymbol{\xi} \in \Xi} \sum_{t=1}^{T} \boldsymbol{q}_t^\top \boldsymbol{x}_t(\boldsymbol{\xi}^t) \\
\text{subject to} \quad & \boldsymbol{f}_1(\boldsymbol{x}_1) \le \mathbf{0} && \forall \boldsymbol{\xi} \in \Xi \\
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Extensions:

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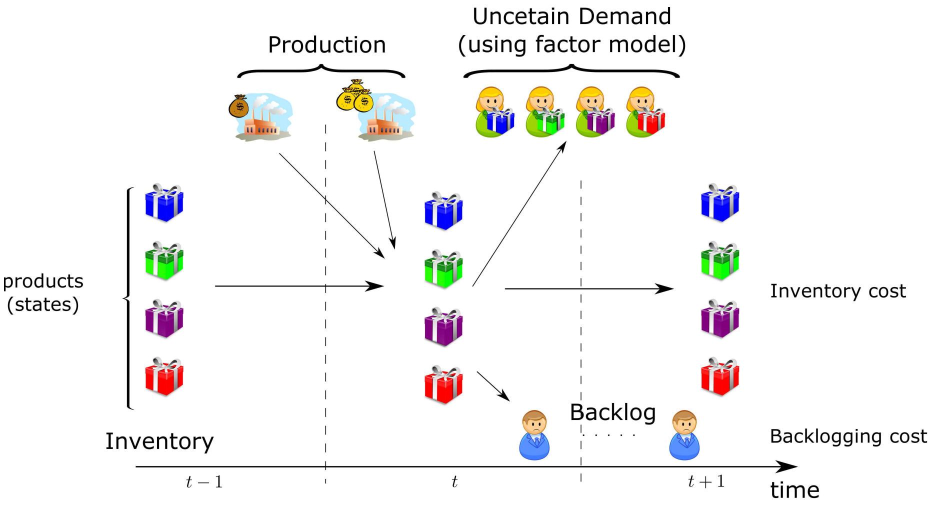
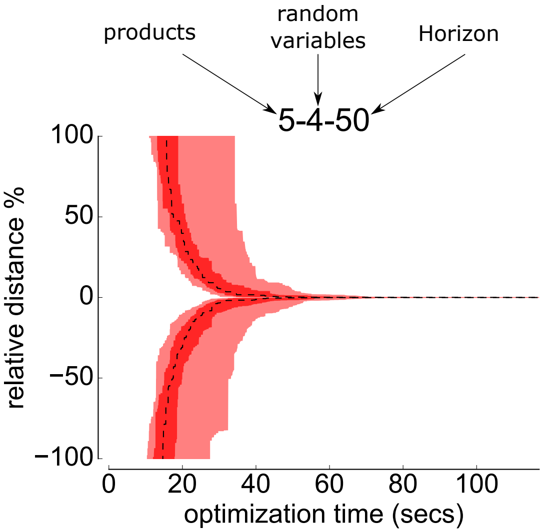
$$

- Random objective function
- Asymptotic convergence guaranties (cost to-go convex but not piecewise linear)

Numerical Results: Inventory Control

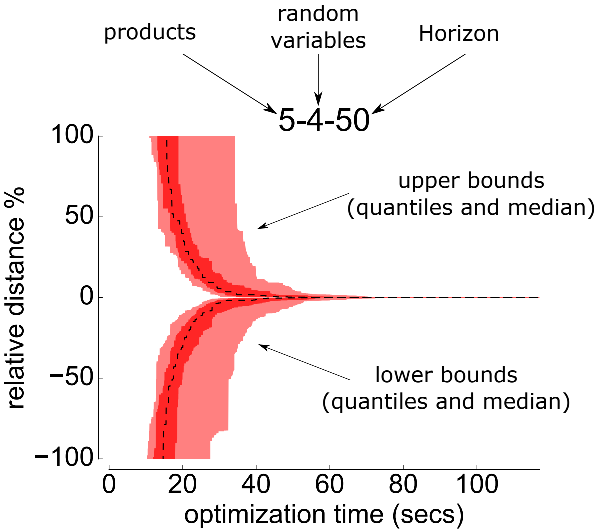


Numerical Results



Results generated using 25 random problem instances

Numerical Results



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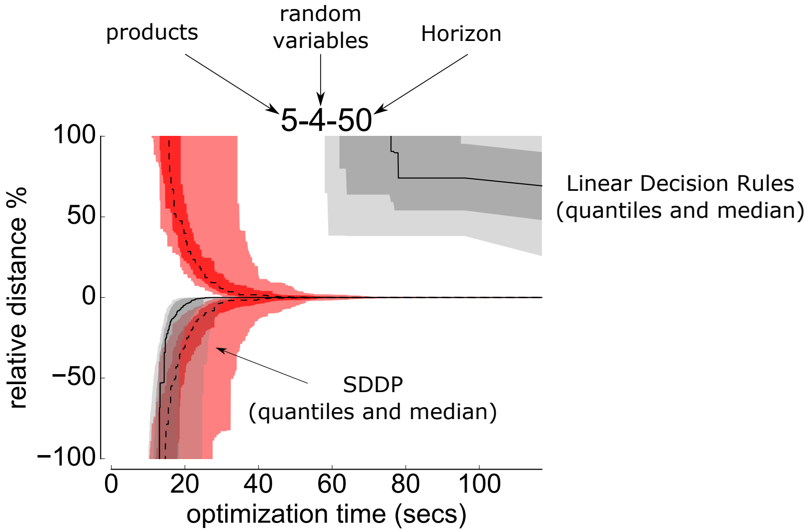


Results generated using 25 random problem instances

Numerical Results: Nested Benders Decomposition

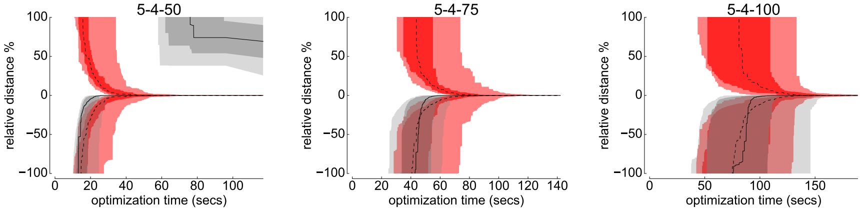
Instance	Trajectories	Runtime	Memory
5-4-3	256	1.3s	18MB
5-4-4	4,096	44.6s	260MB
5-4-5	65,536	924.23s	20.2GB
5-4-6	1,048,576	—	—

- Nested Benders Decomposition is completely impractical for $T > 5$

Numerical Results: RDDP

Scalability w.r.t. horizon $T = \{50, 75, 100\}$

- 5 products (5 states)
- 4 random variables per stage ($2^4 = 16$ scenarios)



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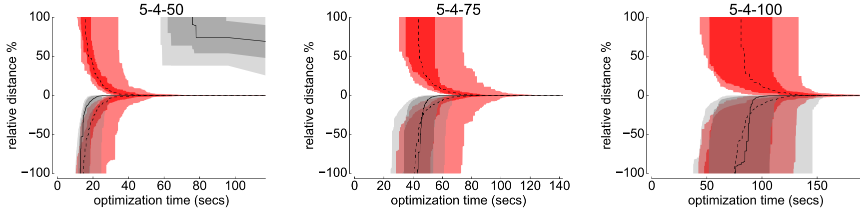
- 5 products (5 states)
- 4 random variables per stage ($2^4 = 16$ scenarios)

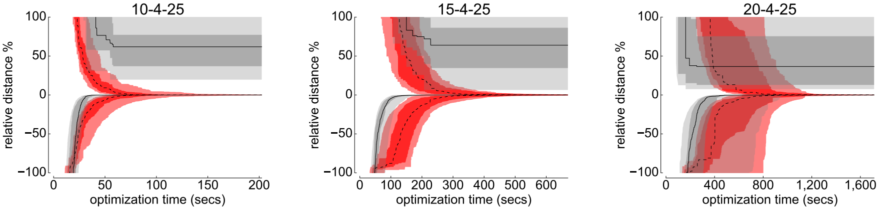


- RDDP scales better than linear decision rules w.r.t. the horizon...
- in addition to converging to the optimal solution

Numerical Results: RDDP

Scalability w.r.t. products $= \{10, 15, 20\}$

- 4 random variables per stage ($2^4 = 16$ scenarios)
- horizon $T = 25$



Numerical Results: RDDP

Scalability w.r.t. products $= \{10, 15, 20\}$

- 4 random variables per stage ($2^4 = 16$ scenarios)
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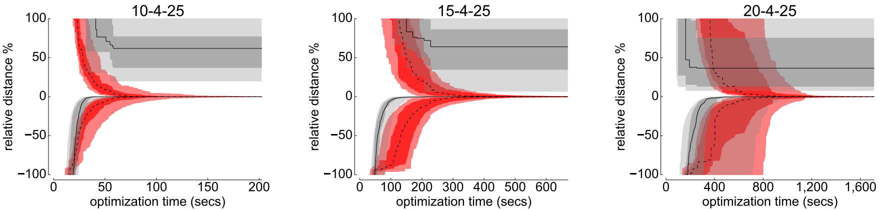
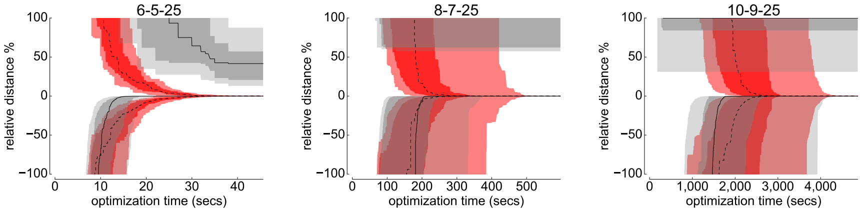


- RDDP does not solve the "curse of dimensionality"
- But, can address problem instances of practical interest ...
- while converging to the optimal solution

Numerical Results: RDDP

Scalability w.r.t. random variables $= \{5, 7, 9\}$

- i.e., scenarios per stage $= \{32, 128, 512\}$
- products $= \{6, 8, 10\}$
- horizon $T = 25$



- Complexity of two-stage problem can affect scalability

Numerical Results: RDDP

Scalability w.r.t. random variables $= \{5, 7, 9\}$

- i.e., scenarios per stage $= \{32, 128, 512\}$
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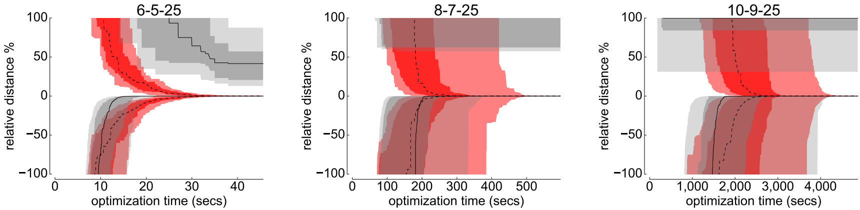


- Complexity of two-stage problem can affect scalability

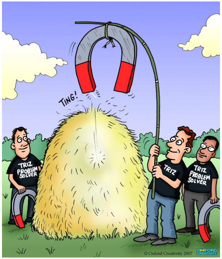
Numerical Results: SDDP

Order	Initial inventories $I_{0p}(\boldsymbol{\xi}^0)$									
	20%		25%		30%		35%		40%	
frequency Δ	Solved	Gap	Solved	Gap	Solved	Gap	Solved	Gap	Solved	Gap
5	70%	18%	60%	20%	40%	20%	20%	72%	0%	100%
7	20%	13%	50%	17%	80%	5%	10%	26%	0%	100%
10	0%	14%	0%	14%	20%	18%	10%	23%	10%	73%

- SDDP can easily miss the optimal solution!

SDDP

RDDP



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