Transformations of Asymptotically AdS Initial Data and Associated Geometric Inequalities

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Initial Data

Consider an initial data set (M, g, k) for the Einstein equations modeling an asymptotically totally geodesic, spacelike hypersurface, in an asymptotically AdS spacetime. This consists of a Riemannian 3-manifold M with asymptotically hyperbolic metric g, and asymptotically vanishing extrinsic curvature k. We define such an initial data set to be **asymptotically AdS hyperbolic** if it possesses an end which is diffeomorphic to $S^2 \times [r_0, \infty)$, and in this region there are coordinates such that $g = g_0 + a$, where $g_0 = \frac{dr^2}{1+r^2} + r^2\sigma$ is the hyperbolic metric with σ the round metric on S^2, and

$$a_{rr} = \frac{\mathbf{m}^r}{r^5} + O_3(r^{-6}), \quad a_{r\alpha} = O_3(r^{-3}), \quad a_{\alpha\beta} = \frac{\mathbf{m}^g_{\alpha\beta}}{r} + O_3(r^{-2}),$$

$$k_{rr} = O_2(r^{-5}), \qquad k_{r\alpha} = O_2(r^{-3}), \qquad k_{\alpha\beta} = \frac{\mathbf{m}^k_{\alpha\beta}}{r} + O_2(r^{-2}).$$

The quantities $\mathbf{m}^g$ and $\mathbf{m}^r$ encode mass through the formula

$$m = \frac{1}{16\pi} \int_{S^2} 3\, Tr_\sigma \mathbf{m}^g + 2\mathbf{m}^r,$$

where the integrand is the so called mass aspect function. Note that this is a slight abuse of terminology since the quantity m is physically the total energy.

Initial data modeling an asymptotically umbilical, spacelike hypersurface, in an asymptotically flat spacetime will be referred to as **asymptotically hyperboloidal**. These type of data are defined by the asymptotics above except that the fall-off for k is replaced by the fall-off for $k - g_0$. In this case the tensor $\mathbf{m}^k$ plays a role in the definition of mass

$$m = \frac{1}{16\pi} \int_{S^2} \left[Tr_\sigma \left(\mathbf{m}^g + 2\mathbf{m}^k \right) + 2\mathbf{m}^r \right].$$

The masses agree for totally umbilic slices.

The Positive Mass Theorem

Theorem (Andersson, Cai, Chrusciel, Herzlich, Galloway, Maerten, Xie, Wang, Zhang)

Let (M, g, k) be a complete asymptotically AdS hyperbolic initial data set satisfying the dominant energy condition. Then $m \geq 0$, and equality holds if and only if (M, g, k) arises from an embedding into anti-de Sitter space.

Most of the methods used to prove this rely on the spinor approach. Here we propose a different strategy based on deformations of the initial data which result in an asymptotically flat structure. The deformations preserve the mass and yield 'sufficiently' nonnegative scalar curvature, so that the result follows from the proof of positivity of the ADM mass given by Schoen and Yau. Thus the PMT in the asymptotically AdS hyperbolic setting is reduced to the PMT in the ADM context, whenever a canonical system of elliptic equations admits a solution.

The deformation procedure consist of three parts.

- The given asymptotically AdS hyperbolic data (M, g, k) is transformed into a time symmetric data set (M_1, g_1) which is also asymptotically AdS hyperbolic, has the same mass, and satisfies the scalar curvature lower bound $R_1 \geq -6$ weakly.
- Following a procedure of Chrusciel and Tod (M_1, g_1) is transformed into an umbilic asymptotically hyperboloidal data set (M_2, g_2, k_2) which again preserves the mass and satisfies $R_2 \geq -6$ weakly.
- Lastly, based on the Schoen/Yau Bondi mass reduction the hyperboloidal data (M_2, g_2, k_2) is deformed into a time symmetric asymptotically flat data set (M_3, g_3), having twice the original mass and nonnegative scalar curvature $R_3 \geq 0$ weakly.
- A trivial example is: [$t = 0$ slice of AdS] $\rightarrow$ [itself] $\rightarrow$ [hyperboloid in Minkowski space] $\rightarrow$ [$t = 0$ slice of Minkowski].

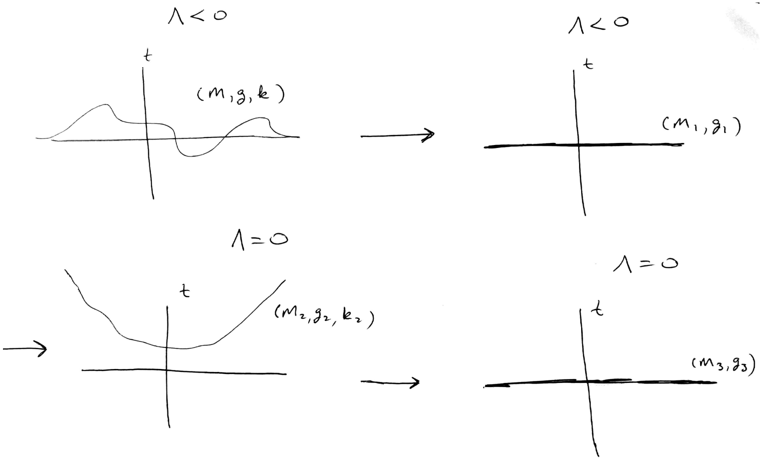

Figure: Deformations

1st Deformation

Let $M_1 = \{t = f(x)\}$ be the graph embedded in the warped product 4-manifold $(M \times \mathbb{R}, g + u^2 dt^2)$, which satisfies the generalized Jang equation

$$\left(g^{ij} - \frac{u^2 f^i f^j}{1 + u^2 |\nabla f|_g^2}\right)\left(\frac{u\nabla_{ij} f + u_i f_j + u_j f_i}{\sqrt{1 + u^2 |\nabla f|_g^2}} - k_{ij}\right) = 0.$$

Then set $g_1 = g + u^2 df^2$ to be the induced metric on this graph. Here u is a positive function that will be chosen appropriately. The purpose of the generalized Jang equation is to impart weakly the desired lower bound $R_1 \geq -6$ for the scalar curvature of g_1. Namely, we have

$$2\mu_1 := R_1 - 2\Lambda = 2(\mu - J(w)) + |\pi - k|_{g_1}^2 + 2|q|_{g_1}^2 - 2u^{-1}\operatorname{div}_{g_1}(uq),$$

where μ_1 is the energy density of the new data (M_1, g_1), π denotes the extrinsic curvature of M in the dual Lorentzian setting $(M_1 \times \mathbb{R}, g_1 - u^2 dt^2)$, and w and q are 1-forms.

In order to preserve the asymptotic geometry, and motivated by the model examples of asymptotically totally geodesic slices in AdS space, the following expansions will be imposed

$$u = \sqrt{1 + r^2} + u_0 + O\left(r^{-1}\right), \qquad f = O\left(r^{-3}\right),$$

where u_0 is a (mass related) constant.

Lemma

If (M, g, k) is asymptotically AdS hyperbolic and the above asymptotics are satisfied, then (M_1, g_1) is asymptotically AdS hyperbolic, and the mass is given by $m_1 = m$.

2nd Deformation

Originally used by Chrusciel/Tod to take a constant mean curvature asymptotically hyperboloidal data set into a maximal asymptotically AdS hyperbolic data set. Here we perform this transformation in the opposite direction to obtain an umbilic asymptotically hyperboloidal data set from a time symmetric asymptotically AdS hyperbolic data set. Namely, consider new data

$$(M_2, g_2) \equiv (M_1, g_1), \qquad k_2 = \sqrt{\frac{-\Lambda}{3}} g_1,$$

with corresponding matter energy and momentum density

$$2\mu_2 = R_2 + (Tr_{g_2} k_2)^2 - |k_2|^2_{g_2} = R_1 - 2\Lambda = 2\mu_1,$$
$$J_2 = \mathrm{div}_{g_2}(k_2 - (Tr_{g_2} k_2) g_2) = 0.$$

Lemma

If (M_1, g_1) is asymptotically AdS hyperbolic then (M_2, g_2, k_2) is asymptotically hyperboloidal with mass given by $m_2 = m_1$.

3rd Deformation

This will yield an asymptotically flat data set from the asymptotically hyperboloidal (M_2, g_2, k_2). Consider a graph $M_3 = \{t = \widetilde{f}(x)\}$ embedded in the product 4-manifold $(M_2 \times \mathbb{R}, g_2 + dt^2)$, so that the induced metric on M_3 is given by $g_3 = g_2 + d\widetilde{f}^2$. Motivated by the model hyperboloidal slices in Minkowski space, the following asymptotics will be imposed on the graph

$$\widetilde{f}(r, \theta, \phi) = \sqrt{1 + r^2} + \mathcal{A} \log r + \mathcal{B}(\theta, \phi) + \hat{f}(r, \theta, \phi),$$

where (θ, ϕ) are coordinates on S^2,

$$\mathcal{A} = 2m_2, \qquad \Delta_\sigma \mathcal{B} = \frac{1}{2}\left[3\,Tr_\sigma \mathbf{m}^g + 2\mathbf{m}^r\right] - \frac{1}{8\pi}\int_{S^2}\left[3\,Tr_\sigma \mathbf{m}^g + 2\mathbf{m}^r\right],$$

and for some $\varepsilon > 0$

$$\hat{f} = O(r^{-1+\varepsilon}).$$

If the classical Jang equation is satisfied

$$\left(g_2^{ij} - \frac{\widetilde{f}^i \widetilde{f}^j}{1 + |\widetilde{\nabla} \widetilde{f}|_{g_2}^2}\right) \left(\frac{\widetilde{\nabla}_{ij} \widetilde{f}}{\sqrt{1 + |\widetilde{\nabla} \widetilde{f}|_{g_2}^2}} - (k_2)_{ij}\right) = 0,$$

then the scalar curvature of the Jang graph (M_3, g_3) enjoys weak nonnegativity through the formula

$$R_3 = 2(\mu_2 - J_2(\widetilde{w})) + |\pi_2 - k_2|_{g_3}^2 + 2|\widetilde{q}|_{g_3}^2 - 2\,\mathrm{div}_{g_3}(\widetilde{q}),$$

where π_2 is the second fundamental form of the graph in the dual Lorentzian setting, and $\widetilde{w}$ and $\widetilde{q}$ are 1-forms. Here $\widetilde{\nabla}$ denotes covariant differentiation with respect to g_2.

Lemma

If (M_2, g_2, k_2) is asymptotically hyperboloidal and satisfies the above asymptotics then (M_3, g_3) is asymptotically flat, and the ADM mass is given by $m_3 = 2m_2$.

At this stage we have an asymptotically flat initial data set (M_3, g_3) which encodes the mass of the original data. If the scalar curvature R_3 is nonnegative, then the (ADM) positive mass theorem would apply to yield the desired result. However this is not exactly the case, since

$$\begin{aligned} R_3 =& 2(\mu - J(w)) + |\pi - k|^2_{g_2} + |\pi_2 - k_2|^2_{g_3} + 2|q|^2_{g_2} + 2|\widetilde{q}|^2_{g_3} \\ & - 2u^{-1}\operatorname{div}_{g_2}(uq) - 2\operatorname{div}_{g_3}(\widetilde{q}). \end{aligned}$$

Nevertheless, the scalar curvature is 'sufficiently' nonnegative to allow the basic strategy of Schoen/Yau to be carried out. Namely if $\psi > 0$ solves

$$\Delta_{g_3}\psi - \frac{1}{8}R_3\psi = 0, \qquad \psi = 1 + \frac{\psi_0}{r} + O(r^{-2}),$$

where ψ_0 is a constant, then $(M_4, g_4) = (M_3, \psi^4 g_3)$ is asymptotically flat with zero scalar curvature. By the AF PMT the mass of the conformal metric is nonnegative $m_4 \geq 0$. Therefore, since $m_4 = m_3 + 2\psi_0$ it remains to show that $\psi_0 \leq -\frac{1}{4}m_3$.

Multiply the zero scalar curvature equations by ψ and integrate by parts

$$
\begin{aligned}
\int_{S_\infty} 4g_3(\psi\nabla\psi,\nu_3) &= \int_{M_3} 4|\nabla_{g_3}\psi|^2 + \frac{1}{2}R_3\psi^2 \\
&\geq \int_{M_3} 4|\nabla_{g_3}\psi|^2 + \left(|\widetilde{q}|^2_{g_3} - \mathrm{div}_{g_3}(\widetilde{q})\right)\psi^2 \\
&\quad + \int_{M_3} \left(|q|^2_{g_2} - u^{-1}\,\mathrm{div}_{g_2}(uq)\right)\psi^2 \\
&\geq 4\pi m_3 + \int_{M_3} 3|\nabla_{g_3}\psi|^2 + \left(|q|^2_{g_2} - u^{-1}\,\mathrm{div}_{g_2}(uq)\right)\psi^2.
\end{aligned}
$$

Since the relation between volume forms is

$$d\omega_{g_3} = \sqrt{1+|\widetilde{\nabla}\widetilde{f}|^2_{g_2}}\,d\omega_{g_2},$$

this motivates the choice

$$u = \psi^2\sqrt{1+|\widetilde{\nabla}\widetilde{f}|^2_{g_2}}$$

which aids with existence and positivity of ψ and guarantees that u follows the desired asymptotics with $u_0 = 2\psi_0 + \mathcal{A}$.

We may now apply the divergence theorem and note that the boundary term vanishes to find the needed inequality

$$-16\pi\psi_0 = \int_{\widetilde{S}_\infty} 4g_3(\psi\nabla_{g_3}\psi, \nu_3) \geq 4\pi m_3.$$

Theorem

Let (M, g, k) be a complete asymptotically AdS hyperbolic initial data set satisfying the dominant energy condition $\mu \geq |J|$. If the coupled Jang system of equations admits a solution with the stated asymptotics, then $m \geq 0$. Moreover $m = 0$ implies that the initial data arise from an embedding into the anti-de Sitter spacetime.

Each equation in the system, when treated independently, is known to admit solutions with the stated asymptotics. However it is an open problem to determine whether the coupled system admits an appropriate solution.

Case of Equality

If $m = 0$ then $\psi_0 = m_1 = m_2 = m_3 = 0$, and in particular the AF PMT shows that (M_3, g_3) is isometric to $(\mathbb{R}^3, \delta)$. It follows that (M_2, g_2, k_2) is a graphical totally umbilical slice of Minkowski space, where the graph has an asymptotic expansion of the form $t = \widetilde{f} = \sqrt{1 + r^2} + B(\theta, \phi) + \hat{f}$. Observe that since g_2 is asymptotically hyperboloidal $(g_2)_{\alpha\beta} = r^2\sigma_{\alpha\beta} + O(r^{-1})$; on the other hand

$$(g_2)_{\alpha\beta} = (g_3)_{\alpha\beta} - \widetilde{f}_\alpha \widetilde{f}_\beta = \delta_{\alpha\beta} - \widetilde{f}_\alpha \widetilde{f}_\beta = r^2\sigma_{\alpha\beta} - \mathcal{B}_\alpha \mathcal{B}_\beta + O(r^{-1+\varepsilon}).$$

Therefore $|\nabla \mathcal{B}|_\sigma = O(r^{-\frac{1}{2}+\frac{\varepsilon}{2}})$, and since $\mathcal{B}$ is independent of r this yields $\mathcal{B} = const$. Next, a result of Choquet-Bruhat shows that if a constant mean curvature spacelike graph in Minkowski space is sufficiently close to the hyperboloid at infinity, then it must be isometric to the hyperboloid. Hence, $\hat{f} \equiv 0$ and $(M_2, g_2) \equiv (M_1, g_1)$ is isometric to hyperbolic 3-space.

The original data (M, g, k) must then arise from an embedding into the AdS spacetime via the graph $t = f(x)$. To see this, observe that

$$g = g_1 - u^2 df^2 = g_0 - u^2 df^2.$$

Moreover since $\psi_0 = 0$ we also have $\psi \equiv 1$, so that together with $\widetilde{f} = \sqrt{1 + r^2} + \mathcal{B}$ it follows that

$$u = \sqrt{1 + r^2}.$$

That is, (M, g) is isometrically embedded into AdS space. Finally, the given extrinsic curvature k agrees with the second fundamental form of this embedding, since the sequence of inequalities and formula for the scalar curvatures implies that

$$|\pi - k|_{g_2} = 0.$$

The Penrose Inequality

In the asymptotically hyperbolic setting the Penrose inequality has two different formulations. For an asymptotically hyperbolic 3-manifold (M, g) one may either consider the area A_0 of the outermost minimal surface, or the area A_2 of the outermost constant mean curvature 2 surface and formulate the inequalities

$$m \geq \sqrt{\frac{A_0}{16\pi}} + \frac{1}{2}\left(\sqrt{\frac{A_0}{4\pi}}\right)^3, \qquad m \geq \sqrt{\frac{A_2}{16\pi}}.$$

In both cases the inequality is conjectured to be saturated only for the corresponding region in the $t = 0$ slice of the Schwarzschild-AdS solution. There does not seem to be much motivation for the first inequality, except for its saturation in Schwarzschild-AdS. The original heuristic arguments of Penrose in the asymptotical flat setting are based on the final state conjecture (of which stability of Kerr is a special case), the Hawking area theorem, and weak cosmic censorship to produce

$$m_i \geq m_f \geq \sqrt{\frac{a_f}{16\pi}} \geq \sqrt{\frac{a_i}{16\pi}}.$$

However in the asymptotically AdS case these arguments break down, since AdS space itself is believed to be unstable. On the other hand, the second inequality is well-motivated as it is modeled on asymptotically hyperboloidal slices in asymptotically flat spacetimes, and thus is derived via the original Penrose arguments where $H = 2$ represents the outermost apparent horizon $\theta_-(S) = H - Tr_S k = 0$ in an umbilic initial data set $k = g$. It does not appear that any results are known for either of these inequality beyond the spherically symmetric case. Here we propose a more general version of the inequality along with a method for reducing it to the asymptotically flat case. The motivation is solely due to the deformation procedure used in the reduction.

Conjecture

Let (M, g, k) by an asymptotically AdS hyperbolic initial data set satisfying the dominant energy condition. Then

$$m \geq \sqrt{\frac{A}{16\pi}},$$

where A is the minimum area required to enclose the outermost mean convex surface ∂M satisfying

$$\theta_+\theta_- = H^2 - (Tr_{\partial M} k)^2 = 4.$$

Moreover, if equality holds then the initial data arise from an embedding into the Schwarzschild-AdS spacetime.

1st Deformation

As with the positive mass theorem, we seek three deformations of the initial data which yield weak nonnegativity of the scalar curvature and preserve the mass in an appropriate sense. In the current context certain aspects of the boundary geometry must also be controlled throughout the process, in particular the mean curvature. The first step is to solve the generalized Jang equation with a warping factor u to be suitably chosen, where the solution f and warping function satisfy the previous asymptotics. According to the lemma this yields a new time symmetric, asymptotically AdS hyperbolic, initial data set (M_1, g_1) with the same mass $m_1 = m$. Moreover, in order to make contact with apparent horizons in the next deformation, geometric boundary conditions are imposed when solving the generalized Jang equation.

Namely

$$H_{\partial M_1} = 2, \qquad q(\nu_{g_1}) = 0 \text{ on } \partial M_1,$$

where

$$q_i = \frac{uf^j}{\sqrt{1+u^2|\nabla f|_g^2}}(\pi_{ij} - k_{ij}).$$

The boundary behavior of ∂M is chosen to facilitate the these boundary conditions. In particular, on level sets of f or when f blows-up

$$H_{\partial M_1} = \frac{H_{\partial M}}{\sqrt{1+(u\partial_{\nu_g} f)^2}}, \quad q(\nu_{g_1}) = -\frac{(u\partial_{\nu_g} f)^2 H_{\partial M}}{\sqrt{1+(u\partial_{\nu_g} f)^2}} + (u\partial_{\nu_g} f) Tr_{\partial M} k.$$

Thus the Neumann-type boundary condition $u\partial_{\nu_g} f = \frac{1}{2} Tr_{\partial M} k$ combined with properties of ∂M yields the desired boundary geometry of the Jang surface.

2nd and 3rd Deformations

The second step in the deformation agrees with that used in the positive mass theorem. According to the lemma we obtain an asymptotically hyperboloidal data set (M_2, g_2, k_2) with preserved mass $m_2 = m_1$. In addition, the boundary conditions give a past apparent horizon boundary $H_{\partial M_2} - Tr_{\partial M_2} k_2 = 0$.

The third and final step in the deformation procedure is a generalization of that which was given for the PMT. It yields a time symmetric, asymtotically flat initial data set (M_3, g_3) as follows. Consider a graph $M_3 = \{t = \widetilde{f}(x)\}$ embedded in the warped product 4-manifold $(M_2 \times \mathbb{R}, g_2 + \widetilde{u}^2 dt^2)$, where $\widetilde{f}$ satisfies the previous asymptotics, and the function $\widetilde{u}$ is nonnegative with asymptotic expansion

$$\widetilde{u} = 1 + \frac{\widetilde{u}_0}{r} + O\left(\frac{1}{r^2}\right),$$

for some constant $\widetilde{u}_0$.

The induced metric on M_3 is given by $g_3 = g_2 + \widetilde{u}^2 d\widetilde{f}^2$. If the generalized Jang equation

$$\left(g_2^{ij} - \frac{\widetilde{u}^2 \widetilde{f}^i \widetilde{f}^j}{1 + \widetilde{u}^2 |\widetilde{\nabla} \widetilde{f}|_{g_2}^2} \right) \left(\frac{\widetilde{u} \widetilde{\nabla}_{ij} \widetilde{f} + \widetilde{u}_i \widetilde{f}_j + \widetilde{u}_j \widetilde{f}_i}{\sqrt{1 + \widetilde{u}^2 |\widetilde{\nabla} \widetilde{f}|_{g_2}^2}} - (k_2)_{ij} \right) = 0$$

is satisfied, then the scalar curvature R_3 of the Jang graph (M_3, g_3) is weakly nonnegative as follows

$$\begin{aligned} R_3 =& 2(\mu_2 - J_2(\widetilde{w})) + |\pi_2 - k_2|_{g_3}^2 + 2|\widetilde{q}|_{g_3}^2 - 2\widetilde{u}^{-1} \operatorname{div}_{g_3}(\widetilde{u}\widetilde{q}) \\ =& 2(\mu - J(w)) + |\pi - k|_{g_2}^2 + |\pi_2 - k_2|_{g_3}^2 + 2|q|_{g_2}^2 + 2|\widetilde{q}|_{g_3}^2 \\ & - 2u^{-1} \operatorname{div}_{g_2}(uq) - 2\widetilde{u}^{-1} \operatorname{div}_{g_3}(\widetilde{u}\widetilde{q}). \end{aligned}$$

When solving this equation the appropriate geometric boundary condition to impose requires the Jang graph to have a minimal surface boundary, $H_{\partial M_3} = 0$. This typically involves blow-up of the Jang equation at the apparent horizon boundary present in M_2.

Lemma

If (M_2, g_2, k_2) is asymptotically hyperboloidal then the Jang metric $g_3 = g_2 + \widetilde{u}^2 d\widetilde{f}^2$ is asymptotically flat, and the mass of the Jang metric is given by $m_3 = 2m_2 + \widetilde{u}_0$.

Consider an inverse mean curvature flow $\{\widetilde{S}_\tau\}$ inside (M_3, g_3) emanating from the minimal boundary $\widetilde{S}_0 = \partial M_3$. Then the monotonicity formula yields

$$
\begin{aligned}
m_3 - \sqrt{\frac{A}{16\pi}} \geq & M_H(\infty) - M_H(0) \\
\geq & c \int_{M_3} R_3 \widetilde{H}_\tau \sqrt{|\widetilde{S}_\tau|} \\
\geq & c \int_{M_3} \left[u^{-1} \operatorname{div}_{g_2}(uq) + \widetilde{u}^{-1} \operatorname{div}_{g_3}(\widetilde{u}\widetilde{q}) \right] \widetilde{H}_\tau \sqrt{|\widetilde{S}_\tau|}
\end{aligned}
$$

where M_H denotes Hawking mass and $\widetilde{H}_\tau$, $|\widetilde{S}_\tau|$ are the mean curvature and area of $\widetilde{S}_\tau$, respectively.

The relation between volume forms is given by

$$d\omega_{g_3} = \sqrt{1 + \widetilde{u}^2 |\widetilde{\nabla}\widetilde{f}|^2_{g_2}}\, d\omega_{g_2},$$

and this motivates the choice

$$\widetilde{u} = \sqrt{\frac{|\widetilde{S}_\tau|}{16\pi}}\widetilde{H}_\tau, \qquad u = \widetilde{u}\sqrt{1 + \widetilde{u}^2 |\widetilde{\nabla}\widetilde{f}|^2_{g_2}}.$$

It follows that

$$m_3 - \sqrt{\frac{A}{16\pi}} \geq c\left[\int_{\overline{S}_0 \cup \overline{S}_\infty} u g_2(q, \nu_{g_2}) + \int_{\widetilde{S}_0 \cup \widetilde{S}_\infty} \widetilde{u} g_3(\widetilde{q}, \nu_{g_3})\right] = 0.$$

The warping factors vanish at the inner boundary, and the chosen asymptotics guarantee that the boundary integrals at infinity also vanish.

It remains to show that $m_3 = m_2$, since previous results give $m_2 = m_1 = m$. To see this observe that

$$
\begin{aligned}
m_3 = M_H(\infty) &= \lim_{\tau\to\infty} \sqrt{\frac{|\widetilde{S}_\tau|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_{\widetilde{S}_\tau} \widetilde{H}_\tau^2\right) \\
&= \lim_{\tau\to\infty} \sqrt{\frac{|\widetilde{S}_\tau|}{16\pi}} \left(1 - \frac{1}{|\widetilde{S}_\tau|}\int_{\widetilde{S}_\tau} \widetilde{u}^2\right) \\
&= - \widetilde{u}_0 .
\end{aligned}
$$

Moreover by the lemma $m_3 = 2m_2 + \widetilde{u}_0$ and hence $m_2 = -\widetilde{u}_0$, so that all masses agree. In fact it suffices to have $m_3 \geq M_H(\infty)$ as this implies that $-\widetilde{u}_0 \leq m_2$.

Theorem

Let (M, g, k) be an asymptotically AdS hyperbolic initial data set with a connected outermost mean convex boundary with $\theta_+\theta_- = 4$, and such that the dominant energy condition $\mu \geq |J|_g$ holds. If the coupled system of equations with stated boundary conditions admits a solution with appropriate asymptotics, then

$$m \geq \sqrt{\frac{A}{16\pi}}$$

where A is the minimum area required to enclose the boundary. If equality is achieved then the initial data arise from an embedding into the Schwarzschild-AdS spacetime.

It should be possible to treat the case of multiple component boundaries by coupling the generalized Jang equations with Bray's conformal flow.

The Case of Equality

In the case of equality we may appeal to the proof in the asymptotically flat case to find that $\widetilde{u} = \sqrt{1 - \frac{2m}{r}}$, and (M_3, g_3), (M_2, g_2, k_2) are respectively isometric to the $t = 0$ slice and a graphical totally umbilic slice of the Schwarzschild spacetime. Furthermore it can be shown that $\widetilde{f} = \widetilde{f}(r)$ arises from the ODE

$$\widetilde{f}' = \frac{r}{\left(1 - \frac{2m}{r}\right)\sqrt{r^2 + 1 - \frac{2m}{r}}}.$$

We then have

$$u = \widetilde{u}\sqrt{1 + \widetilde{u}^2 |\widetilde{\nabla}\widetilde{f}|^2_{g_2}} = \sqrt{r^2 + 1 - \frac{2m}{r}},$$

which is the warping factor of the Schwarzschild-AdS spacetime. It is straightforward to see that $(M_1, g_1 = g_2 = g_3 - \widetilde{u}^2 d\widetilde{f}^2)$ is then isometric to the $t = 0$ slice of Schwarzschild-AdS. The remainder of the proof follows the arguments in the PMT.

Mass-Angular Momentum Inequalities

In the asymptotically flat case precise inequalities relating the ADM mass and angular momentum may also be derived from Penrose's heuristic arguments. In order to accomplish this, additional hypotheses are needed to ensure that total angular momentum is conserved throughout the evolution. This usually entails the assumption of axisymmetry and some condition on the matter fields, vacuum being the simplest such condition. The following sequence of inequalities between initial and final states is then obtained

$$m_i \geq m_f \geq \sqrt{|\mathcal{J}_f|} = \sqrt{|\mathcal{J}_i|}.$$

This inequality has been established by Dain and others in the maximal case for simply connected axisymmetric asymptotically flat initial data with two ends if the nonnegative energy condition is satisfied $\mu \geq 0$. The inequality is saturated precisely for the $t = 0$ slice of extreme Kerr. As with the Penrose inequality these heuristic arguments do not carry over to the AH setting.

Mass-Angular Momentum Inequalities in the AH Setting

In analogy with Kerr-AdS we may propose the inequality

$$m \geq \frac{1}{3\sqrt{6}} \left[\sqrt{\left(1+\frac{\mathcal{J}^2}{m^2}\right)^2 + \frac{12\mathcal{J}^2}{m^2}} + 2\left(1+\frac{\mathcal{J}^2}{m^2}\right) \right] * \left[\sqrt{\left(1+\frac{\mathcal{J}^2}{m^2}\right)^2 + \frac{12\mathcal{J}^2}{m^2}} - \left(1+\frac{\mathcal{J}^2}{m^2}\right) \right]^{1/2},$$

which is saturated for extreme Kerr-AdS black holes. This is unwieldy and we seek a more simple inequality implied by this one. Namely, if m_{ext} denotes the right-hand side then $m_{ext} > |\mathcal{J}|/m$ and $m_{ext} > |\mathcal{J}|$ unless $\mathcal{J} = 0$. Therefore this implies

$$m \geq \sqrt{|\mathcal{J}|}, \quad m \geq |\mathcal{J}| \text{ BPS Bound: Chrusciel/Maerten/Tod.}$$

However it is not expected that any proper black hole solution saturates this bound, since within the Kerr-AdS family such a state would result in a naked singularity.

In contrast to the previous discussion, here the initial data (M, g, k) shall have two asymptotic ends. One end M^{+}_{end}, from which the mass m arises, is designated asymptotically AdS hyperbolic, whereas the other end M^{-}_{end} is either asymptotically AdS hyperbolic or asymptotically cylindrical. It will be assumed that M is simply connected and that the data are are axisymmetric. The later means that a subgroup of the group of isometries is isomorphic to $U(1)$, and that all quantities associated with the initial data are invariant under this $U(1)$ action. In particular, if $\eta = \partial_\phi$ denotes the Killing field which generates the symmetry, then $\mathfrak{L}_\eta g = \mathfrak{L}_\eta k = 0$. Axisymmetry allows for a simple definition of angular momentum via the following Komar integral

$$\mathcal{J} = \frac{1}{8\pi} \int_S (k_{ij} - (Tr_g k) g_{ij}) \nu^i_g \eta^j,$$

where S is any surface which is homologous to the 'sphere at infinity' associated with M^{+}_{end}. This definition is well-defined (independent of the choice of S) as long as $J_i \eta^i = 0$.

As before we seek three deformations of the initial data such that M_{end}^{+} is transformed into an asymptotically flat end, and M_{end}^{-} is transformed into an asymptotically flat/asymptotically cylindrical end if it was originally asymptotically AdS hyperbolic/asymptotically cylindrical. Furthermore, the topology of M as well as physical quantities such as mass and angular momentum will be preserved. The 1st deformation produces a new data set (M_1, g_1, k_1) as follows. Consider a graph $M_1 = \{t = f(x)\}$ embedded in the warped product stationary 4-manifold

$$(M \times \mathbb{R}, g + 2Y_i dx^i dt + (u^2 - |Y|_{g_1}^2)dt^2),$$

with induced metric

$$(g_1)_{ij} = g_{ij} + f_i Y_j + f_j Y_i + (u^2 - |Y|_{g_1}^2) f_i f_j.$$

The new extrinsic curvature is defined to be the second fundamental form of the $t = 0$ slice in the dual Lorentzian setting and is given by

$$(k_1)_{ij} = \frac{1}{2u}\left(\overline{\nabla}_i Y_j + \overline{\nabla}_j Y_i\right),$$

Motivated by the structure of the Kerr-AdS spacetime, it will be assumed that Y has a single component in the Killing direction

$$Y^i\partial_i := (g_1)^{ij}Y_j\partial_i = Y^\phi\partial_\phi.$$

This condition also guarantees that g_1 is a Riemannian metric. Thus, the first deformation is characterized by three functions (u, Y^ϕ, f), which are axisymmetric and will be chosen appropriately below.

In order to have a well-defined angular momentum after deformation, we must have

$$\mathrm{div}_{g_1} k_1(\eta) = 0.$$

This is a linear elliptic equation for Y^ϕ, if u and f are independent of Y^ϕ. The total angular momentum will be preserved if the following asymptotics hold

$$Y^\phi = -\frac{2\mathcal{J}}{r^3} + O(r^{-4}) \ \text{ in } \ M^+_{end}, \quad Y^\phi \to 0 \ \text{ in } \ M^-_{end}.$$

Next we choose f to satisfy the Jang-type equation

$$g^{ij}\left(\frac{u\nabla_{ij}f+u_if_j+u_jf_i}{\sqrt{1+u^2|\nabla f|_g^2}}-k_{ij}\right)=0,$$

with asymptotics given as before. Then

$$2\mu_1=R_1-|k_1|_{g_1}^2-2\Lambda=2(\mu-J(v))+|k-\pi|_g^2+2u^{-1}\operatorname{div}_{g_1}(uQ),$$

for some 1-form Q. If the dominant energy condition is valid, then $\mu_1\geq 0$ holds weakly in the sense that this quantity differs from a nonnegative function by a divergence term. It remains to choose u. This will require knowledge of further deformations, and thus will be explained at the end.

Lemma

The initial data set (M_1, g_1, k_1) is axially symmetric and maximal $Tr_{g_1}k_1=0$, and preserves the asymptotic geometry of (M, g, k) as well as the mass and angular momentum $m_1=m$, $\mathcal{J}_1=\mathcal{J}$.

We will now proceed to the second deformation. The goal is to construct a constant mean curvature (CMC) asymptotically hyperboloidal data set, denoted (M_2, g_2, k_2), from the maximal asymptotically AdS data (M_1, g_1, k_1). In order to accomplish this define

$$(M_2, g_2) \equiv (M_1, g_1), \qquad\qquad (k_2)_{ij} = (k_1)_{ij} + \sqrt{\frac{-\Lambda}{3}}(g_1)_{ij}.$$

Observe that $Tr_{g_2} k_2 = \sqrt{-3\Lambda}$ and

$$2\mu_2 = R_2 + (Tr_{g_2} k_2)^2 - |k_2|^2_{g_2} = R_1 - |k_1|^2_{g_1} - 2\Lambda = 2\mu_1,$$
$$J_2 = \mathrm{div}_{g_2}(k_2 - (Tr_{g_2} k_2) g_2) = J_1.$$

In particular, this shows that the condition for a well-defined angular momentum is satisfied

$$J_2(\eta) = J_1(\eta) = 0.$$

Lemma

The initial data set (M_2, g_2, k_2) is CMC and axially symmetric with two ends, one designated asymptotically hyperboloidal $(M_2)^+_{end}$ and the other $(M_2)^-_{end}$ either asymptotically hyperboloidal or asymptotically cylindrical depending on whether $(M_1)^-_{end}$ is asymptotically AdS hyperbolic or asymptotically cylindrical. Furthermore $m_2 = m_1$ and $\mathcal{J}_2 = \mathcal{J}_1$.

The third and final deformation is similar to the first, modulo the asymptotics.

Lemma

The initial data set (M_3, g_3, k_3) is maximal and axially symmetric with two ends, one designated asymptotically flat $(M_3)^+_{end}$ and the other $(M_3)^-_{end}$ either asymptotically flat or asymptotically cylindrical depending on whether $(M_2)^-_{end}$ is asymptotically hyperboloidal or asymptotically cylindrical. Furthermore $m_3 = m_2$ and $\mathcal{J}_3 = \mathcal{J}_2$.

Combining the deformations leads to

$$\begin{aligned}2\mu_3 =& R_3 - |k_3|^2_{g_3}\\ =& 2(\mu - J(v)) + |k-\pi|^2_g + |k_2 - \pi_2|^2_{g_2}\\ &+ 2u^{-1}\operatorname{div}_{g_1}(uQ) + 2\widetilde{u}^{-1}\operatorname{div}_{g_3}(\widetilde{u}\widetilde{Q}).\end{aligned}$$

We are now in a position to choose u and $\widetilde{u}$. This will be done by appealing to aspects of the proof of the mass-angular momentum inequality in the AF case. Since M_3 is simply connected and axisymmetric, a result of Chrusciel shows that it is diffeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and admits a global system of Brill (cylindrical) coordinate system (ρ, z, ϕ) in which the metric takes the simple form

$$g_3 = e^{-2U+2\alpha}(d\rho^2 + dz^2) + e^{-2U}\rho^2(d\phi + A_\rho d\rho + A_z dz)^2.$$

Brill Scalar Curvature Formula

The structure of the metric in Brill coordinates implies a nice formula for the scalar curvature

$$2e^{-2U+2\alpha}R_3 = 8\Delta U - 4\Delta_{\rho,z}\alpha - 4|\nabla U|^2 - \rho^2 e^{-2\alpha}\left(\partial_z A_\rho - \partial_\rho A_z\right)^2,$$

where $\Delta_{\rho,z} = \partial_\rho^2 + \partial_z^2$. Moreover, since $J_3(\eta) = 0$ and M_3 is simply connected, a twist potential exists

$$d\chi = \star\left(\eta \wedge k_3(\eta)\right) = *\left(\eta \wedge d\eta\right)|_{M_3},$$

and yields the lower bound

$$R_3 \geq (R_3 - |k_3|_{g_3}^2) + 2\frac{e^{6U-2\alpha}}{\rho^4}|\nabla\chi|^2.$$

By integrating the scalar curvature we find that

$$m_3 \geq \underbrace{\frac{1}{32\pi}\int_{\mathbb{R}^3}\left(4|\nabla U|^2 + \frac{e^{4U}}{\rho^4}|\nabla\chi|^2\right)dx}_{\mathcal{E}(U,\chi)} + \frac{1}{16\pi}\int_{\mathbb{R}^3}(R_3 - |k_3|^2_{g_3})dx.$$

The first integral $\mathcal{E}(U,\chi)$ is the reduced harmonic energy of a singular map $\mathbb{R}^3 \to \mathbb{H}^2$ given by $x \to (\mathbf{U}(x), \chi(x))$ where $\mathbf{U} = -\log\rho + U$. The second integral may be estimated from below with the derived scalar curvature formula

$$m_3 - \mathcal{E}(U,\chi) \geq \frac{1}{8\pi}\int_{M_2}\frac{e^{\widetilde{U}}\sqrt{1+\widetilde{u}^2|\widetilde{\nabla}\widetilde{f}|^2_{g_2}}}{u}\operatorname{div}_{g_2}(uQ)$$
$$+\frac{1}{8\pi}\int_{M_3}\frac{e^{\widetilde{U}}}{\widetilde{u}}\operatorname{div}_{g_3}(\widetilde{u}\widetilde{Q}).$$

This then motivates the choices

$$\widetilde{u} = e^{\widetilde{U}}, \qquad u = \widetilde{u}\sqrt{1+\widetilde{u}^2|\widetilde{\nabla}\widetilde{f}|^2_{g_2}}.$$

Minimize the Reduced Energy

The divergence theorem reduces the right-hand side to boundary terms in the two asymptotic ends, which can be shown to vanish if appropriate asymptotics for the solutions are imposed. Moreover, the negative curvature of the target space may be exploited to minimize the harmonic energy among all such maps with the same angular momentum. In particular, convexity of the energy along geodesic deformations shows that the unique minimum is achieved for the extreme Kerr harmonic map with fixed angular momentum. Therefore

$$\mathcal{E}(U,\chi) \geq \mathcal{E}(U_k,\chi_k) = \sqrt{|\mathcal{J}_3|}.$$

Theorem

Let (M, g, k) be a simply connected axially symmetric initial data set satisfying the dominant energy condition $\mu \geq |J|_g$ and $J(\eta) = 0$, having two ends, one designated asymptotically AdS hyperbolic and the other either asymptotically AdS hyperbolic or asymptotically cylindrical. If the system of Jang-type equations admits a solution with appropriate asymptotics then

$$m \geq \sqrt{|\mathcal{J}|}.$$

By working backwards through the deformations, if equality is achieved then $Y^{\phi} = 0$ and hence $\mathcal{J} = 0$, a contradiction. Furthermore these results may be extended to include contributions from the electromagnetic field.