Chernoff's distribution and differential equations of parabolic and Airy type

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joint work with:
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Analytical characterization of distribution of $Z = \text{argmax}\{W(t) - t^2 : t \in \mathbb{R}\}$?

Figure: Herman Chernoff

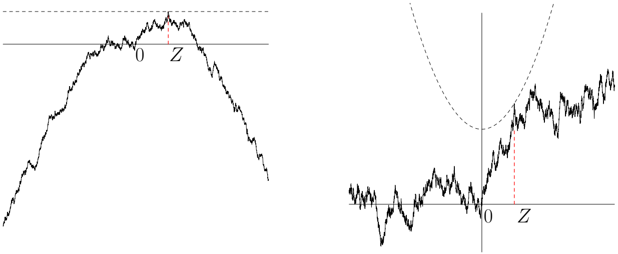

Figure: Left: Z and $t \mapsto W(t) - t^2$, right: Z and $t \mapsto W(t)$.

Chernoff's heat equation (Chernoff (1964))

- Define $u(s, x)$ by

$$u(s, x) = \mathbb{P}\left\{W(t) > t^2 \text{ for some } t > s | W(s) = x\right\}.$$

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If $x < s^2$:

$$\begin{aligned} u(s,x) = \mathbb{P}\left\{W(t) > t^2 \text{ for some } t > s+\epsilon | W(s) = x\right\} + \\ + \mathbb{P}\left\{W(t) > t^2 \text{ for some } t \in (s, s+\epsilon], \right. \\ \left. W(t) \leq t^2, \forall t > s+\epsilon | W(s) = x\right\} \end{aligned}$$

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Hence $u(s, x) = \mathbb{P}\left\{W(t) > t^2 \text{ for some } t > s | W(s) = x\right\}$ satisfies the heat equation:

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Distribution location of maximum (Chernoff (1964))

- Let $M_h = max_{t\in[s-h,s]} W(t)$. Then:

$$\max_{t\in[s-h,s]}\{W(t)-t^2\} = M_h - s^2 + O(h)$$

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Using the spatial homogeneity of Brownian motion we get, taking $\widetilde{W}(t) = W(t) - M_h + s$,

$$\mathbb{P}\left\{\max_{t\geq s}\left\{W(t) - t^2\right\} > M_h - s^2 \;\middle|\; W(s), W(s-h), M_h\right\}$$

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&= \mathbb{P}\left\{\max_{t \geq s} \left\{\widetilde{W}(t) - t^2\right\} > 0 \;\middle|\; \widetilde{W}(s) = W(s) - M_h + s^2\right\} \\
&= u\left(s, W(s) - M_h + s^2\right) \\
&= u(s, s^2)\,(= 1) + \{W(s) - M_h\}\,\partial_2 u(s, s^2) + O_p(h).
\end{aligned}$$

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- Similarly:

$$\mathbb{P}\left\{\max_{t\le s-h}\left\{W(t)-t^2\right\}\ge M_h-(s-h)^2 \;\middle|\; W(s-h),M_h\right\}$$
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- Conclusion:

$$\mathbb{P}\{Z\in[s-h,s]\}$$
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$$=\tfrac{1}{2}h\partial_2 u(s,s^2)\partial_2 u(-s,s^2),\; h\downarrow 0.$$

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Then the density f_Z *of* $Z = \text{argmax}\{W(x) - x^2\}$ *is given by:*

$$f_Z(s) = \tfrac{1}{2}\partial_2 u(-s, s^2)\partial_2 u(s, s^2).$$

where $u(s,x)$ *solves the heat equation:*

$$\frac{\partial}{\partial s}u(s,x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u(s,x).$$

subject to:

$$u(s,x) = 1, \quad x \geq s^2, \qquad u(s,x) \to 0, \quad x \to -\infty.$$

Computation of density

Original computations of this density were based on numerically solving Chernoff's heat equation.
But (Groeneboom (1984)):

$$\partial_2 u(-s, s^2) \sim c_1 \exp\left\{-\tfrac{2}{3}s^3 - cs\right\}, \; s \to \infty,$$

where $c \approx 2.9458$ and $c_1 \approx 2.2638$. This entails that a numerical solution of this partial differential equation on a grid will not give a really accurate solution!

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Chernoff's result: the density of $\text{argmax}_t\{W(t) - t^2\}$ is given by:

$$f_Z(s) = \tfrac{1}{2}u_2(s)u_2(-s),$$

where

$$\begin{aligned} u_2(s) = \lim_{x \uparrow 0} \frac{\partial}{\partial x} u(s,x) &= \lim_{x \uparrow 0} \frac{\partial}{\partial x} Q^{(s,x)}\left\{X_t \geq 0, \;\text{ for some } t \geq s\right\} \\ &= -\lim_{x \uparrow 0} \frac{\partial}{\partial x} Q^{(s,x)}\left\{X_t < 0, \, \forall t \geq s\right\} \\ &= -\lim_{x \uparrow 0} \frac{\partial}{\partial x} Q^{(s,x)}\left\{\tau_0 = \infty\right\}, \end{aligned}$$

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Cameron-Martin-Girsanov

(1) under $P^{(s,x)}$, $\{X_t = W_t : t \geq s\}$ is standard Brownian motion with $X_s = x$,

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Cameron-Martin-Girsanov: $Q^{(s,x)} << P^{(s,x)}$ on $\{\mathcal{F}_t : t \geq s\}$, where $\mathcal{F}_t = \sigma\{X_u : u \in [s,t]\}$ and $\left.\frac{dQ^{(s,x)}}{dP^{(s,x)}}\right|_{\mathcal{F}_t} = Z_t$, and where

$$Z_t = \exp\left\{2\int_s^t X_u\,du - 2\big(tX_t - sX_s\big) - \tfrac{2}{3}\big(t^3 - s^3\big)\right\}, \quad t \geq s.$$

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Hence, if $x < 0$, $s < t$, and τ_0 is the first time X_t hits zero:

$$\begin{aligned} &Q^{(s,x)}\{\tau_0 \in dt\} \\ &= \exp\left\{2sx - \tfrac{2}{3}\big(t^3 - s^3\big)\right\} \\ &\qquad \cdot E^{P^{(s,x)}}\left\{\exp\left\{2\int_s^t X_u\,du\right\} \,\Big|\, \tau_0 = t\right\} P^{(s,x)}\{\tau_0 \in dt\}. \end{aligned}$$

Airy functions

Let, for $\lambda > 0$, u_λ be the unique non-negative solution of the boundary problem

$$\frac{1}{2}u''(x) - \big(\lambda - 2x\big)u(x) = 0,\ x < 0,\ \lim_{x\uparrow 0} u(x) = 1,\ u(x) \leq 1,\ x \leq 0.$$

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$$Y_t = e^{-\int_{v=s}^t \left(\lambda - 2X_v\right)\,dv} u_\lambda(X_t),$$

where X_t is standard Brownian motion, starting at $x < 0$ at time s. Y_t is a local martingale (Ito's formula).

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where X_t is standard Brownian motion, starting at $x < 0$ at time s. Y_t is a local martingale (Ito's formula). Hence, if $t > s$.

$$E^{P^{(s,x)}} e^{-\int_{v=s}^{\tau_0}\big(\lambda - 2X_v\big)\,dv} = E^{P^{(s,x)}} Y_{\tau_0} = Y_s = u_\lambda(x),$$

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Conclusion:

$$\int_{t\in(s,\infty)} e^{-\lambda(t-s)} E^{P^{(s,x)}} \left\{ e^{\int_{v=s}^{t} 2X_v \, dv} \;\middle|\; \tau_0 = t \right\} P^{(s,x)} \{\tau_0 \in dt\}$$

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So we can compute:

$$\begin{aligned} &Q^{(s,x)}\{\tau_0 \in dt\} \\ &= \exp\left\{2sx - \tfrac{2}{3}\left(t^3 - s^3\right)\right\} \\ &\qquad \cdot E^{P^{(s,x)}}\left\{e^{\int_{v=s}^{t} 2X_v\,dv} \,\middle|\, \tau_0 = t\right\} P^{(s,x)}\{\tau_0 \in dt\}, \end{aligned}$$

where $Q^{(s,x)}$ is the probability measure of $\{W_t - t^2 : t \geq s\}$, starting at x at time s.

Inverse Laplace transforms

Inversion of the Laplace transform along the imaginary axis:

$$Q^{(s,x)}\{\tau_0 < \infty\} = \int_{t\in(s,\infty)} Q^{(s,x)}\{\tau_0 \in dt\}$$

$$= \frac{e^{2sx+\frac{2}{3}s^3}}{2\pi} \int_{v=-\infty}^{\infty} \frac{\mathrm{Ai}\left(2^{-1/3}iv - 4^{1/3}x\right)}{\mathrm{Ai}\left(2^{-1/3}iv\right)} \int_{t=0}^{\infty} e^{itv-\frac{2}{3}(s+t)^3}\, dt\, dv.$$

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So we would be done if we can deal with the properties of the integral in the last line, since Chernoff's function $u(s,x)$ satisfies

$$u(s,x) = Q^{(s,x)}\left\{\tau_0 < \infty\right\}.$$

Taking the special case $s = 0$, we get:

$$\frac{1}{\pi}\int_{t=0}^{\infty} e^{ivt-\frac{1}{3}t^3}\,dt = \mathrm{Hi}(iv),$$

where Hi denotes Scorer's function Hi.

Difficulties of direct approach

Scorer's function Hi has the asymptotic expansion, as $|z| \to \infty$,

$$\mathrm{Hi}(z) \sim -\frac{1}{\pi z} \sum_{k=0}^{\infty} \frac{(3k)!}{k!(3z^3)^k}, \qquad |\mathrm{ph}(-z)| < \tfrac{2}{3}\pi - \delta$$

for $\delta > 0$ arbitrarily small, where $\mathrm{ph}(-z)$ denotes the phase of $-z$.

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for $\delta > 0$ arbitrarily small, where $\mathrm{ph}(-z)$ denotes the phase of $-z$. This means that

$$\int_{v=-\infty}^{\infty} \left\{ \frac{\mathrm{Ai}\left(2^{-1/3} iv - 4^{1/3} x\right)}{\mathrm{Ai}\left(2^{-1/3} iv\right)} \int_{t=0}^{\infty} e^{itv - \frac{2}{3}(s+t)^3} \, dt \right\} dv$$

has a non-integrable integrand if $x = 0$.

Difficulties of direct approach

Scorer's function Hi has the asymptotic expansion, as $|z| \to \infty$,

$$\mathrm{Hi}(z) \sim -\frac{1}{\pi z} \sum_{k=0}^{\infty} \frac{(3k)!}{k!(3z^3)^k}, \qquad |\mathrm{ph}(-z)| < \tfrac{2}{3}\pi - \delta$$

for $\delta > 0$ arbitrarily small, where $\mathrm{ph}(-z)$ denotes the phase of $-z$. This means that

$$\int_{v=-\infty}^{\infty} \left\{ \frac{\mathrm{Ai}\big(2^{-1/3} iv - 4^{1/3} x\big)}{\mathrm{Ai}\big(2^{-1/3} iv\big)} \int_{t=0}^{\infty} e^{itv - \frac{2}{3}(s+t)^3}\, dt \right\} dv$$

has a non-integrable integrand if $x = 0$. Whereas in fact:

$$\lim_{x \uparrow 0} \int_{v=-\infty}^{\infty} \frac{\mathrm{Ai}\big(2^{-1/3} iv - 4^{1/3} x\big)}{\mathrm{Ai}\big(2^{-1/3} iv\big)} \int_{t=0}^{\infty} e^{itv - \frac{2}{3}(s+t)^3}\, dt\, dv$$

$$= \lim_{x \uparrow 0} Q^{(s,x)} \{\tau_0 < \infty\} = 1.$$

Indirect approach

For this reason the limit

$$Q^{(s,x)}\{\tau_0 = \infty\} = \lim_{t\to\infty} Q^{(s,x)}\{\tau_0 > t\}$$

was computed by first determining the transition density

$$Q^{(s,x)}\left\{X_t^{\partial} \in dy\right\},\ t > s,\ x, y < 0,$$

of the process X_t^{∂}, which is the process X_t, killed when reaching 0.

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of the process X_t^{∂}, which is the process X_t, killed when reaching 0. Details of this computation in appendix of Groeneboom (1989), yielding:

$$Q^{(s,x)}\left\{\tau_0=\infty\right\}$$
$$=c_{s,x}\int_{v=-\infty}^{\infty}e^{-isv}\,\frac{\mathrm{Ai}(i\xi)\mathrm{Bi}\big(i\xi-4^{1/3}x\big)-\mathrm{Ai}\big(i\xi-4^{1/3}x\big)\mathrm{Bi}\big(i\xi\big)}{\mathrm{Ai}\big(i\xi\big)}\,dv,$$

where $\xi=2^{-1/3}v$ and $c_{s,x}=4^{-1/3}\exp\{(2/3)s^3+2sx\}$.

Since we also know:

$$Q^{(s,x)}\{\tau_0 = \infty\}$$
$$= 1 - \frac{e^{2sx+\frac{2}{3}s^3}}{2\pi} \int_{v=-\infty}^{\infty} \frac{\mathrm{Ai}\big(2^{-1/3}iv - 4^{1/3}x\big)}{\mathrm{Ai}\big(2^{-1/3}iv\big)} \int_{t=0}^{\infty} e^{itv-\frac{2}{3}(s+t)^3}\, dt\, dv,$$

we must have the analytic relation

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$$= c_{s,x} \int_{v=-\infty}^{\infty} e^{-isv} \frac{\mathrm{Ai}(i\xi)\mathrm{Bi}\big(i\xi - 4^{1/3}x\big) - \mathrm{Ai}\big(i\xi - 4^{1/3}x\big)\mathrm{Bi}\big(i\xi\big)}{\mathrm{Ai}\big(i\xi\big)}\, dv.$$

Since we also know:

$$
\begin{aligned}
&Q^{(s,x)}\{\tau_0=\infty\}\\
&=1-\frac{e^{2sx+\frac23 s^3}}{2\pi}\int_{v=-\infty}^{\infty}\frac{\mathrm{Ai}\bigl(2^{-1/3}iv-4^{1/3}x\bigr)}{\mathrm{Ai}\bigl(2^{-1/3}iv\bigr)}\int_{t=0}^{\infty}e^{itv-\frac23(s+t)^3}\,dt\,dv,
\end{aligned}
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\begin{aligned}
&\Bigl(Q^{(s,x)}\{\tau_0=\infty\}=\Bigr)\\
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&=c_{s,x}\int_{v=-\infty}^{\infty}e^{-isv}\frac{\mathrm{Ai}(i\xi)\mathrm{Bi}\bigl(i\xi-4^{1/3}x\bigr)-\mathrm{Ai}\bigl(i\xi-4^{1/3}x\bigr)\mathrm{Bi}\bigl(i\xi\bigr)}{\mathrm{Ai}\bigl(i\xi\bigr)}\,dv.
\end{aligned}
$$

If we could prove this relation analytically, we wouldn't need to introduce the process X_t^{∂}, and we would not need the complicated computations in the appendix of Groeneboom (1989).

Lemma (PDE for first expression)

Let the function $f : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ *be defined by*

$$f(s,x) = \frac{1}{2\pi} \int_{v=-\infty}^{\infty} \frac{\mathrm{Ai}\big(i\xi - 4^{1/3}x\big)}{\mathrm{Ai}\big(i\xi\big)} \int_{t=0}^{\infty} e^{itv - \frac{2}{3}(s+t)^3} \, dt \, dv,$$

where $\xi = 2^{-1/3} v$.

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Moreover $0 \leq f(s,x) \leq e^{-2sx - \frac{2}{3}s^3}$, $\lim_{s\to\infty} f(s,x) = 0$, *and*

$$\lim_{x\uparrow 0} f(s,x) = e^{-\frac{2}{3}s^3}, \qquad \lim_{x\to-\infty} e^{2sx} f(s,x) = 0, \qquad s \in \mathbb{R}.$$

Lemma (PDE for second expression)

Let the function $g : \mathbb{R} \times (-\infty, 0] \to \mathbb{R}$ *be defined by*

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Moreover,

$$\lim_{x \uparrow 0} g(s,x) = 0, \qquad \lim_{x \to -\infty} e^{2sx} g(s,x) = e^{-\frac{2}{3}s^3}, \qquad s > 0.$$

Theorem

(i) *Let f and g be as in the preceding lemmas. Then:*

$$f(s,x) = e^{-2sx - \frac{2}{3}s^3} - g(s,x), \quad s \in \mathbb{R}, \quad x \leq 0.$$

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Proof.

$$f(s,x) + g(s,x) - e^{-2sx-\frac{2}{3}s^3} = 0, \qquad s \in \mathbb{R},\, x \leq 0.$$

□

Proof of $h(s,x) = f(s,x) + g(s,x) - e^{-2sx-\frac{2}{3}s^3} \equiv 0.$

One can show $h(s,0) = 0$, $s \in \mathbb{R}$, and:

$$\lim_{x \to -\infty} h(s,x) = 0, \forall s \in \mathbb{R}, \quad \text{and} \quad \lim_{s \to \infty} h(s,x) = 0, \forall x < 0.$$

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Consider an infinite rectangle $R_c = \{(s,x) : s \geq c, x \leq 0\}$, $c \in \mathbb{R}$. Suppose h attains a strictly positive maximum over R_c at an interior point $(s_0, x_0) \in R_c^0$. Then $\partial_1 h(s_0, x_0) = 0$. Hence

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This implies: $\partial_2^2 h(s_0,x_0) = -4x_0 h(s_0,x_0) > 0,$ since $x_0 < 0$ and $h(s_0,x_0) > 0$.

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This implies: $\partial_2^2 h(s_0,x_0) = -4x_0 h(s_0,x_0) > 0$, since $x_0 < 0$ and $h(s_0,x_0) > 0$. Contradiction with $h(s_0,x_0)$ is maximum.

Proof of $h(s,x) = f(s,x) + g(s,x) - e^{-2sx-\frac{2}{3}s^3} \equiv 0$.

So a strictly positive maximum or strictly negative minimum over R_c can only be attained on the line $s = c$.

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Suppose that a strictly positive maximum is attained at the point (c, x_0), where $x_0 < 0$. Then we must have: $\partial_1 h(c, x_0) \le 0$, implying by the partial differential equation for h:

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contradicting the assumption that h attains its maximum on the line $s = c$ at the point (c, x_0).

Conclusion: h is identically zero on R_c. Since the argument holds for all $c \in \mathbb{R}$, we get that the function h is identically zero on $\mathbb{R} \times (-\infty, 0]$.

Theorem (Groeneboom (1984),Daniels and Skyrme (1985))

The probability density f of the location of the maximum of the process $t \mapsto W(t) - t^2$, $t \in \mathbb{R}$, is given by

$$f(s) = \tfrac{1}{2} g(s) g(-s),$$

where

$$g(s) = \frac{1}{2^{2/3}\pi} \int_{-\infty}^{\infty} \frac{e^{-ius}}{\mathrm{Ai}(i2^{-1/3}u)}\, du.$$

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Distribution of the maximum itself:

Janson, Louchard, and Martin-Löf (2010), Groeneboom (2010) Groeneboom and Temme (2011) and Groeneboom, Lalley, and Temme (2013) (joint density of max and argmax).

Henry Daniels

Density of $Z = \text{argmax}\{W(t) - t^2,\ t \in \mathbb{R}\}$

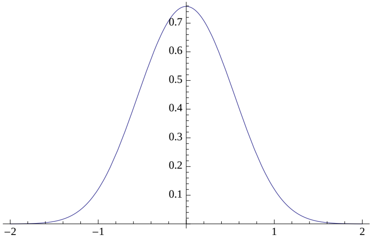

Figure: The density f_Z of the location of the maximum Z of $W(t) - t^2,\ t \in \mathbb{R}$.

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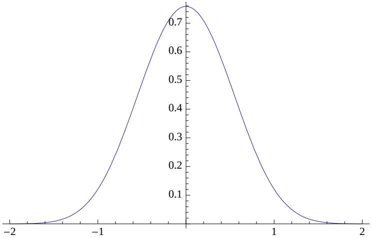


Figure: The density f_Z of the location of the maximum Z of $W(t) - t^2,\ t \in \mathbb{R}$.

Also:

$$\text{var}(Z) = \tfrac{1}{3}\mathbb{E}\max_t \left\{W(t) - t^2\right\},$$

as proved in Groeneboom (2011) and Janson (2013), and (not using the relation with Airy functions) in Pimentel (2014).

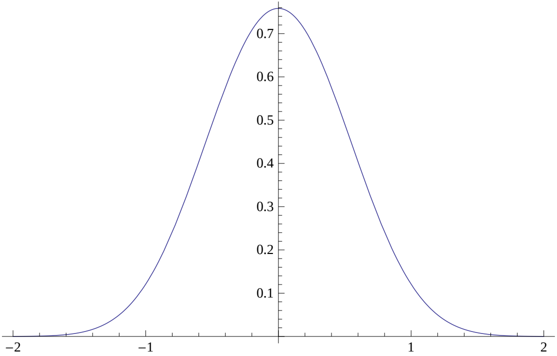
The density can be computed by two lines in Mathematica:

```
In[1]:= f[x_] := (1 / (2 * Pi)) * 2^(1 / 3) *
          Re[NIntegrate[Exp[-I * u * x] / AiryAi[I * 2^(-1 / 3) * u], {u, -10, 10}]]

In[2]:= g[x_] := (1 / 2) * f[x] * f[-x]

In[3]:= Plot[g[x], {x, -2, 2}]
```

Out[3]=



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