A Proof of the Shareshian–Wachs Conjecture

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Stanley's chromatic symmetric function X_G

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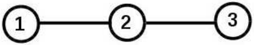
Example.



$$\begin{aligned} X_G &= 6(x_1x_2x_3 + x_1x_2x_4 + \cdots) + (x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_3^2x_1 + \cdots) \\ &= 6e_3 + e_1e_2 - 3e_3 \\ &= 3e_3 + e_1e_2. \end{aligned}$$

Indifference graphs

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Note. The original Stanley–Stembridge conjecture was seemingly more general; Guay-Paquet reduced it to the statement above.

Schur function expansion of X_G

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Conjecture (Shareshian–Wachs, 2011). $\mathrm{ch}\,\rho = \omega X_G$ where ρ is the dot action on the cohomology of a regular semisimple Hessenberg variety and ω is the involution on symmetric functions corresponding to tensoring with the sign representation. In fact, ωX_G naturally decomposes into summands corresponding to each cohomology group H^{2d} separately.

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Our main result is a proof of this conjecture (2015). Independently and almost simultaneously, Guay-Paquet gave a proof using completely different methods (Hopf algebras).

Classification of indifference graphs

Let $\mathbf{m} = (m_1, \dots, m_{n-1})$ be a weakly increasing sequence of integers such that $i \leq m_i \leq n$ for all i (sometimes called a **Hessenberg function**).

Example. If $n = 3$ then $\mathbf{m} \in \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$.

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Let $G(\mathbf{m})$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and in which i and j are adjacent if $i < j \leq m_i$.

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Fact (implicit in earlier literature, explicit in Shareshian–Wachs). $G(\mathbf{m})$ is an indifference graph, and every indifference graph is isomorphic to some $G(\mathbf{m})$.

(Type A) Hessenberg varieties

A **complete flag** in an n-dimensional (complex) vector space V is a nested sequence of subspaces $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = V$ such that $\dim F_i = i$ for all i. The set of all complete flags forms a space called the **complete flag variety**.

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Definition (De Mari–Procesi–Shayman). Let $s \in \mathfrak{gl}_n(V)$. Then the **Hessenberg variety** $\mathscr{H}(\mathbf{m}, s)$ is defined by

$$\mathscr{H}(\mathbf{m}, s) := \{\text{complete flags such that } sF_i \subseteq F_{m_i} \text{ for all } i.\}$$

If s is diagonalizable, we say $\mathscr{H}(\mathbf{m}, s)$ is **semisimple**. If the Jordan blocks of s have distinct eigenvalues, we say $\mathscr{H}(\mathbf{m}, s)$ is **regular**.

The dot action

Diagonal matrices form a torus T that acts on $\mathscr{H}(\mathbf{m}, s)$.

Hessenberg varieties have no odd-dimensional cohomology, so in particular, **Goresky–Kottwitz–MacPherson** theory tells us that the T-equivariant cohomology can be completely described by a combinatorial object called the **moment graph**.

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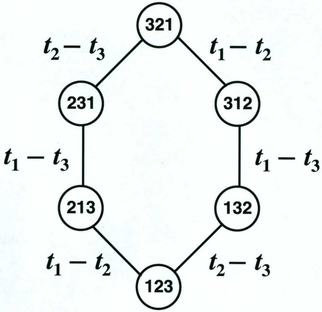
Ordinary cohomology is a quotient of equivariant cohomology. Tymoczko defined an action, the **dot action**, of the symmetric group on the cohomology of a regular semisimple Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$. The action depends only on $\mathbf{m}$ and not on the choice of regular semisimple s.

The moment graph

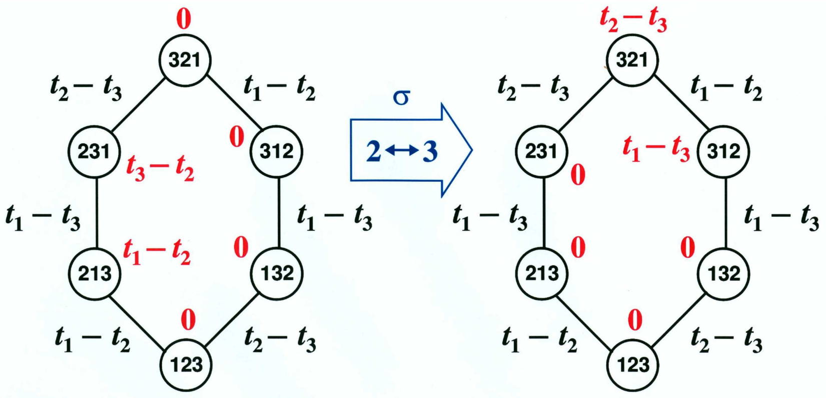
Example on right: $n = 3$, $m_i = i + 1$.

- The vertices are the permutations of $\{1, 2, \ldots, n\}$.
- A transposition (i, j) is **admissible** if $i < j \leq m_i$. For $m_i = i + 1$, these are the adjacent transpositions.
- Two permutations are adjacent if they differ by an admissible transposition on **positions**.
- An edge is labeled with $t_i - t_j$ where i and j are the transposed **numbers**.



The dot action

- An **equivariant cohomology class** c is an assignment of a polynomial $c(w)$ in the t's to each vertex w such that polynomials on adjacent vertices differ by a multiple of the edge label.
- If $\sigma \in S_n$ then $(\sigma c)(w)$ is obtained by taking $c(\sigma^{-1}w)$ (where $\sigma^{-1}w$ means letting σ^{-1} act on the *numbers* of w) and then applying σ to the *subscripts* of the t's.
- Equivariant cohomology classes comprise a free module over $\mathbb{C}[t_1, \ldots, t_n]$. Write down matrices for the above representation with respect to some basis, and then take the constant terms of the entries to get the dot action on the cohomology.


0
321
$t_2 - t_3$
$t_1 - t_2$
231
312
0
$t_3 - t_2$
$t_1 - t_3$
$t_1 - t_3$
$t_1 - t_2$
0
213
132
0
$t_1 - t_2$
$t_2 - t_3$
123
σ
2 ↔ 3
$t_2 - t_3$
321
$t_2 - t_3$
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Linchpin of proof

Theorem. Let λ be a partition of n. Let $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ be a Young subgroup of S_n. Let s be a regular matrix with Jordan type λ. Then the dimension of the subspace of H^{2d} fixed by S_λ under the dot action on a regular *semisimple* Hessenberg variety equals the Betti number β_{2d} of $\mathscr{H}(\mathbf{m}, s)$.

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Standard fact. The dimensions of the above fixed subspaces are the coefficients of the **monomial symmetric function** expansion. In particular, knowing all these numbers completely specifies the representation of S_n.

Therefore the above theorem reduces the computation of the dot action to the computation of regular (but not necessarily semisimple) Hessenberg varieties.

The combinatorial part of the proof

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Corollary. The Betti numbers of a regular Hessenberg variety form a palindromic sequence. (Follows from a theorem of Shareshian and Wachs. Note that regular Hessenberg varieties are not smooth, and the corollary is not true if s is not regular. This corollary has since been generalized to other types by Precup.)

Reciprocity

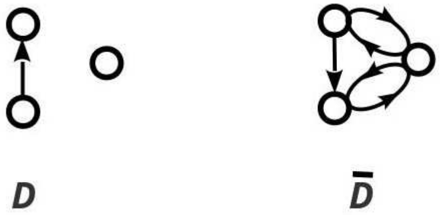
Loosely speaking, a reciprocity theorem yields a combinatorial interpretation of the involution ω.

Our reciprocity theorem yields a combinatorial interpretation of the **monomial** symmetric function expansion of $\omega X_G(t)$.

The key idea is to consider **directed graphs** and to associate a certain quasisymmetric function $\Xi_D(t)$ with any directed graph D that enumerates **ordered path covers** of D according to the number of ascents.

1. For any poset P, $\Xi_{D(P)}(t) = X_{G(P)}(t)$, where $u \to v$ in $D(P)$ iff $u < v$ in P and $G(P)$ is the incomparability graph of P.
2. If $\bar{D}$ is the **complement** of D then $\omega \Xi_D(t) = \Xi_{\bar{D}}(t)$.

Reciprocity example ($t = 1$)



$$\Xi_D = X_G = m_{21} + 6m_{111}$$
$$\omega\Xi_D = 4m_3 + 5m_{21} + 6m_{111} = \Xi_{\bar{D}}$$

Betti Numbers of Regular Hessenberg Varieties

Theorem (Tymoczko). A regular Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$ of Jordan type λ is paved by affines. The nonempty cells are in bijection with tableaux T of shape λ such that k is immediately to the left of j iff $k \leq m_j$. The dimension of a nonempty cell is the sum of

1. the number of pairs $k < i$ in T such that
   - i and k are in the same row, with i left of k,
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The paths in an ordered path cover of $\overline{D(\mathbf{m})}$ correspond naturally to rows in T. But showing that the number of pairs $k < i$ in Tymoczko's theorem coincides with the number counted by $\Xi_{\overline{D(\mathbf{m})}}(t)$ requires a combinatorial argument.

The geometric part of the proof

There is a natural monodromy action of the symmetric group on the equivariant cohomology of a regular semisimple Hessenberg variety. We show that this monodromy action is the same as Tymoczko's dot action, thereby relating the S_λ invariants to a space of **local invariant cycles**.

Work of Beilinson–Bernstein–Deligne on perverse sheaves then implies that there is a **surjection** from the cohomology of regular Hessenberg varieties to the space of local invariant cycles.

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Note. Abe–Harada–Horiguchi–Masuda had previously proved the relationship between the invariants and the cohomology in the special case of regular nilpotent s, but not using a local invariant cycle theorem.