Macdonald polynomials and chromatic quasisymmetric functions

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Outline

- Chromatic functions $X_D(x; t)$

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- Loose ends

Chromatic functions

Dyck paths

- A *Dyck path of order n* is a path that from $(0,0)$ to (n,n) using steps
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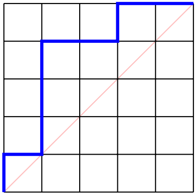

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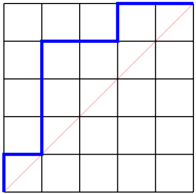

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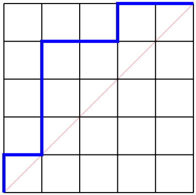

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- $|\mathcal{D}_n| = \frac{1}{2n+1}\binom{2n}{n}$, the nth Catalan number.

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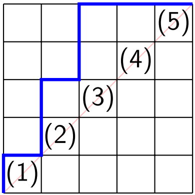

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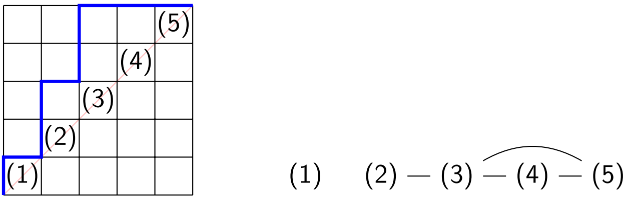

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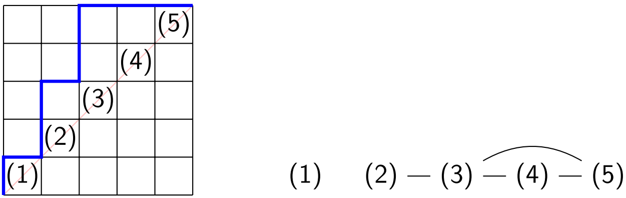

- In fact, this is a bijection from $\mathcal{D}_n$ to incomparability graphs of natural unit interval orders.

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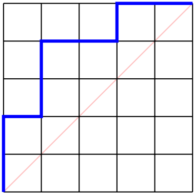

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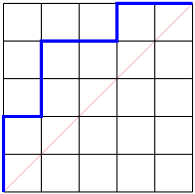

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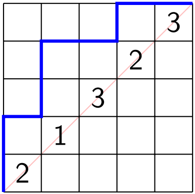

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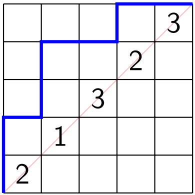

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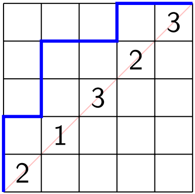

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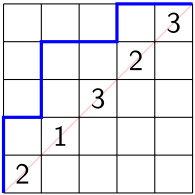

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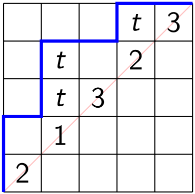

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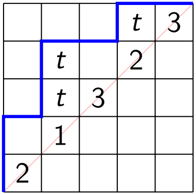

$\to t^3 x_1 x_2^2 x_3^2$

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- A brief aside on "symmetric functions..."

A crash course in symmetric functions

- $\Lambda =$ the ring of symmetric functions.
  - These are power series f in variables $x_1, x_2, \ldots$ that are invariant under the action

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- Λ is often considered in terms of its (many) linear bases.
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- Let's define a few of these.

Classical symmetric function bases

For a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$,

$$m_\lambda = \sum_{i_1 \neq i_k} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}$$

$$p_n = \sum_i x_i^n$$

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Many more . . . .

Schur functions

Schur functions are the unique basis s_μ satisfying

$$\begin{aligned} s_\mu &\in \text{span}\{m_\lambda : \lambda \leq \mu\} \\ s_\mu\big|_{m_\mu} &= 1 \\ \langle s_\lambda, s_\mu \rangle &= 0 \text{ if } \lambda \neq \mu \end{aligned}$$

for

- $<$ an extension of the *dominance order*, and
- $\langle -, - \rangle$ the *Hall inner product*.

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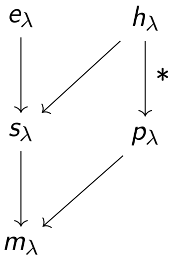
- Is f positive in other bases?

Positivity graph



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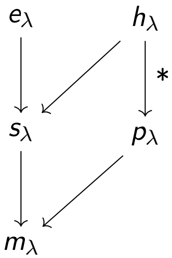

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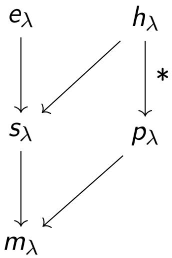

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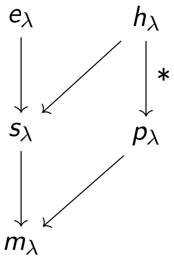

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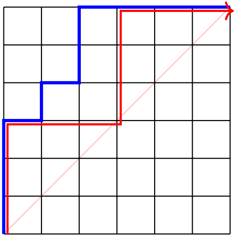
- proven e positive for "one-bounce" paths [HP17].



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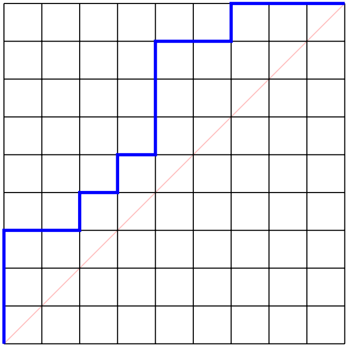

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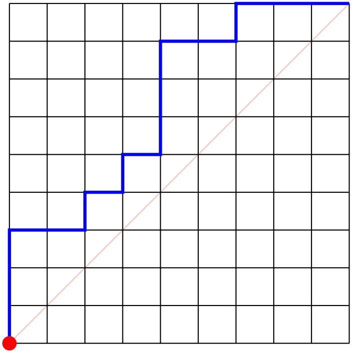

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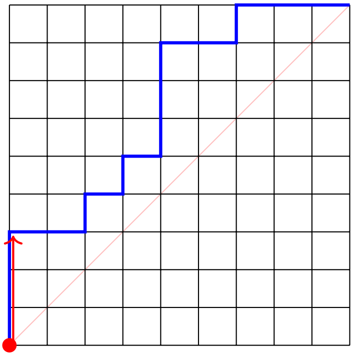

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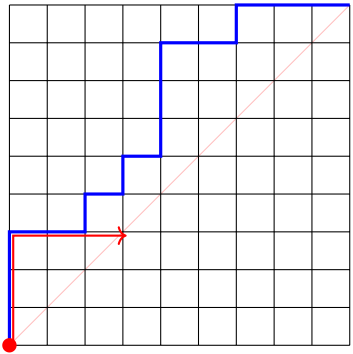

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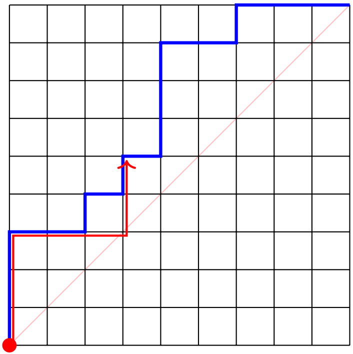

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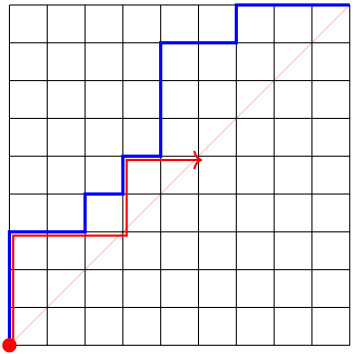

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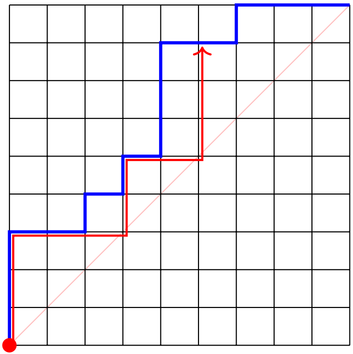

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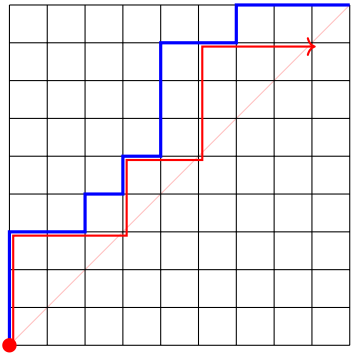

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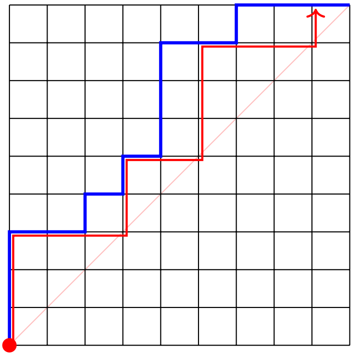

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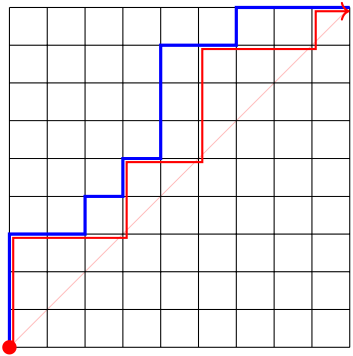

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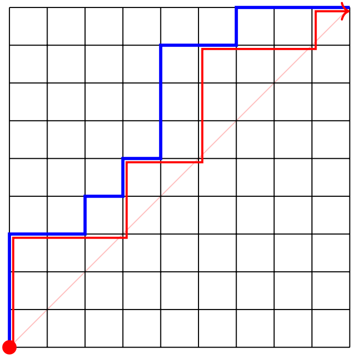

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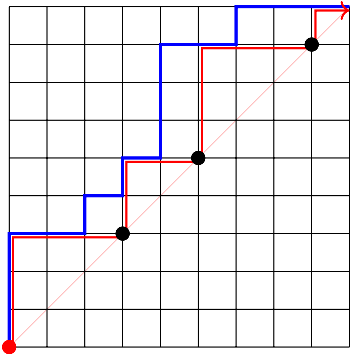

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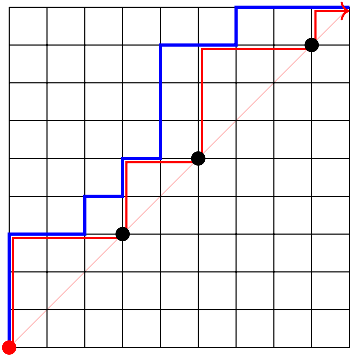

- This is the *bounce path* (Haglund).
- The *bounce length* is 3.

A bounce characterization of height

Theorem [KOP02]

The bounce length of a Dyck path is equal to the height of its Hessenberg ideal.

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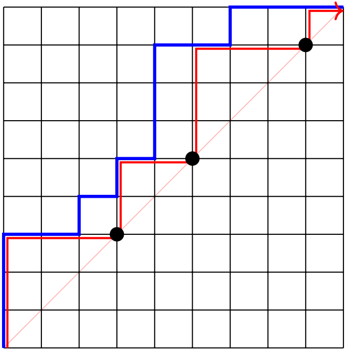

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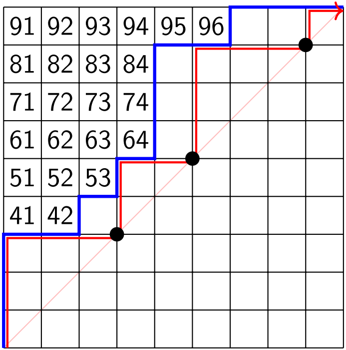

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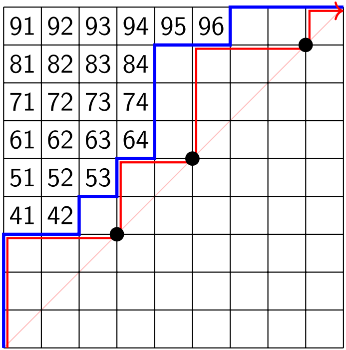

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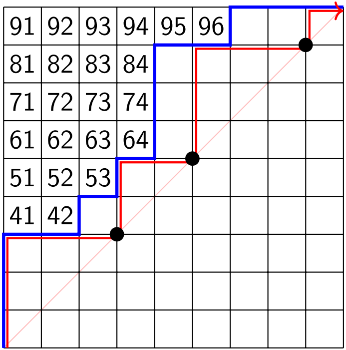

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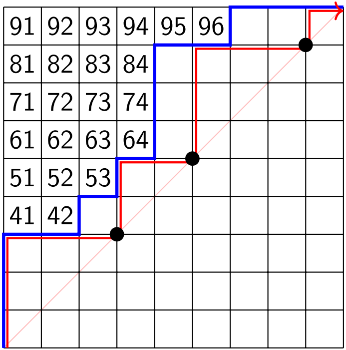

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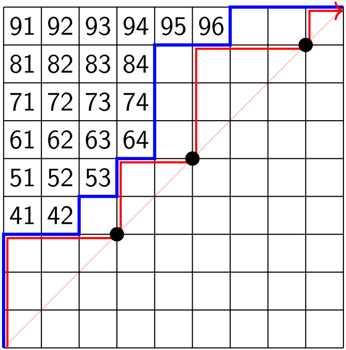

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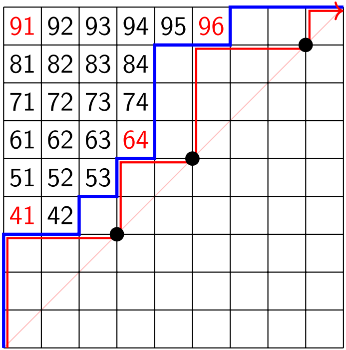

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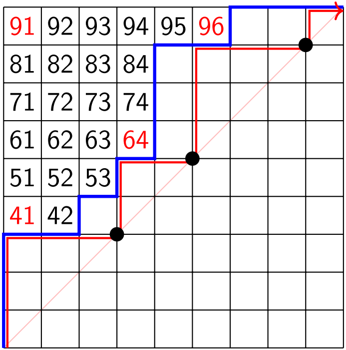

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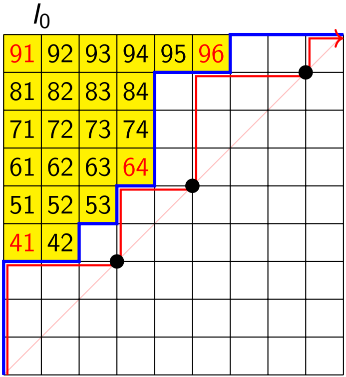

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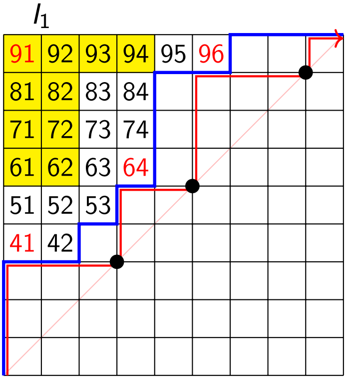

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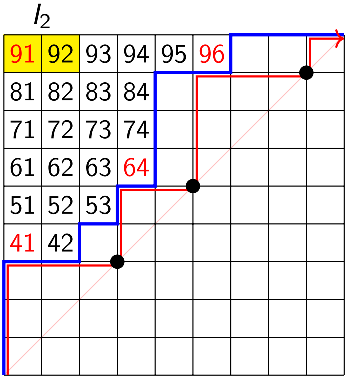

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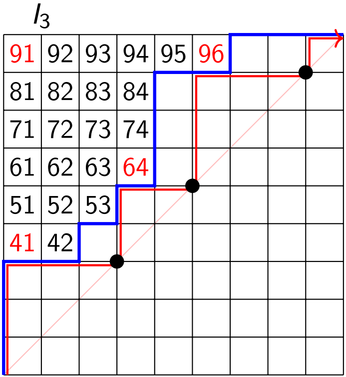

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LLT polynomials

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  - Fundamental to symmetric function theory!

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- Scary formula incoming . . . .

Power sum expansion

Corollary

$$\omega LLT_D(x;t) = \sum_{\lambda \vdash n} \frac{(t-1)^{n-\ell(\lambda)} p_\lambda}{z_\lambda} \sum_{\sigma \in \widetilde{\mathcal{N}}_\lambda(D)} t^{\mathrm{inv}_D(\sigma)}$$

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- Can this relationship be pushed further?

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- Macdonald showed that a unique basis $P_\mu \in \Lambda_{\mathbb{Q}(q,t)}$ existed with the properties:

$$P_\mu \in \mathrm{span}\{m_\lambda : \lambda \leq \mu\}$$
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  generalizing Schur functions to a q, t-inner product.
- He obtained the *integral forms* J_μ by "clearing denominators."
- A combinatorial formula for J_μ was found in [HHL05] involving proper fillings.

A sample integral Macdonald polynomial

$$J_{2,1} = \left(-2qt^4 + 5qt^3 - t^4 - 3qt^2 + t^3 - qt + 3t^2 + q - 5t + 2\right) m_{1,1,1} + \left(-qt^4 + 2qt^3 - qt^2 + t^2 - 2t + 1\right) m_{2,1}$$

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- Not m positive.
- What could a "combinatorial" formula look like?

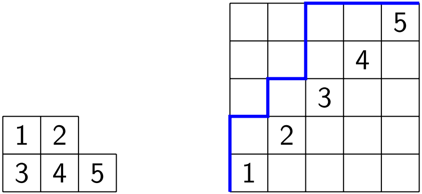
Maybe something like this

Theorem [HHL08]

$$J_{\mu'}(x;q,t) = \sum_{\substack{\sigma:\mu\to\mathbb{Z}_{>0} \\ \sigma \text{ non-attacking}}} x^\sigma q^{\mathrm{maj}(\sigma,\mu)} t^{n(\mu')-\mathrm{inv}(\sigma,\mu)}$$
$$\times \prod_{\substack{u\in\mu \\ \sigma(u)=\sigma(\mathrm{down}_\mu(u))}} \left(1 - q^{\mathrm{leg}_\mu(u)+1} t^{\mathrm{arm}_\mu(u)+1}\right)$$
$$\times \prod_{\substack{u\in\mu \\ \sigma(u)\neq\sigma(\mathrm{down}_\mu(u))}} (1-t)\,.$$

Partitions to Dyck paths

- Given a partition μ, we form a Dyck path D_μ as illustrated.
  - $\#$ squares above i inside $D = \#$ cells after i in reading order before we return to i's column in μ.



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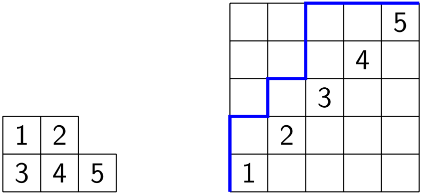

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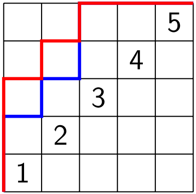
1	2	
3	4	5



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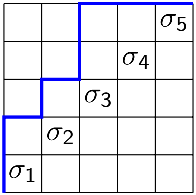
The coefficients are in $\mathbb{Z}[q,t,t^{-1}]$ but we can show that each term is in $\mathbb{Z}[q,t]$ in e.g. the Schur basis.

Example

Say $\mu = (3, 2)$, so $\mu' = (2, 2, 1)$.

σ_1	σ_2	
σ_3	σ_4	σ_5



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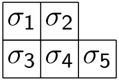

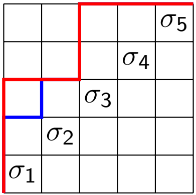

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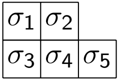

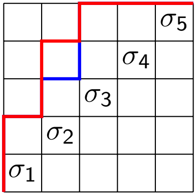

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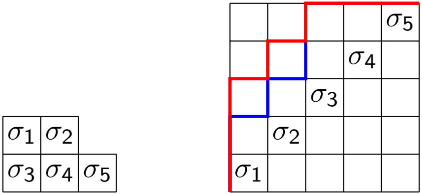

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- Let's see the Schur expansion formula.

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Integral form tableaux (IFT)

- $T \in \mathrm{IFT}_{\lambda,\mu}$ is a bijection $T : \lambda \to [n]$ such that
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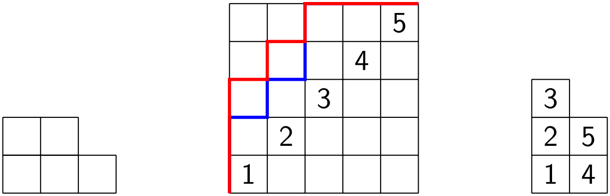
- An example for $\mu = (3, 2)$, $\lambda = (2, 2, 1)$:



Schur expansion

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$$J_{\mu'}(x; q, t)\Big|_{s_\lambda} = \sum_{T \in \mathrm{IFT}_{\lambda,\,\mu}} \mathrm{wt}(T)$$

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- Summing these weights and multiplying by $(1-t)^2$, we get

$$J_{3,1}(x;q,t)\big|_{s_{2,2}} = (1-t)^2(q-t)(1-qt)(1+qt).$$

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- We get similar expansions for p basis.
- All formulas specialize to integral form Jack polynomials.
- Don't know how to manage cancellation yet.

Loose ends

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$$\Lambda_{\mathbb{Q}(q,t)} \to \mathbb{Q}(q,t)[x_1, \ldots, x_n]$$

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- $\mathcal{E}_\mu(x;q,t)$ also have a combinatorial formula [HHL08].
- We can write $\mathcal{E}_\gamma$ as a sum of certain *nonsymmetric chromatic functions*.

Nonsymmetric chromatic functions

- Start with a *partial Dyck path* from $(0, k)$ to (n, n).

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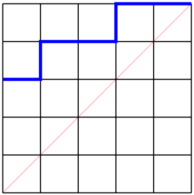

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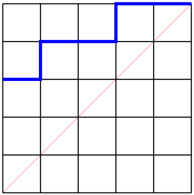

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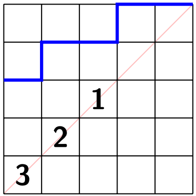

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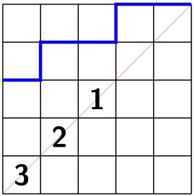

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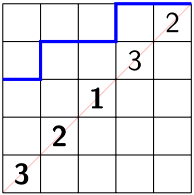

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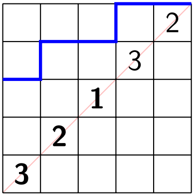

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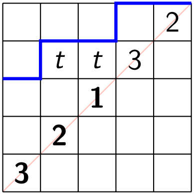

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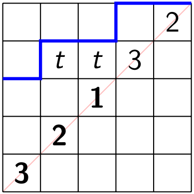

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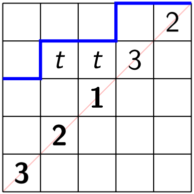
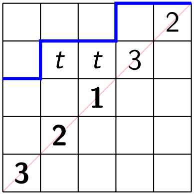


$\rightarrow t^2 x_2 x_3$

More on nonsymmetric chromatic functions



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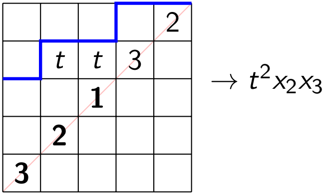

- These are similar to *partial Dyck path characters* [CM15].

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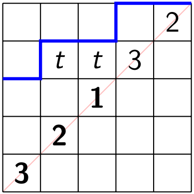

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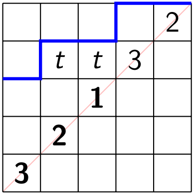

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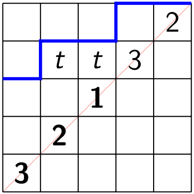

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- These are similar to *partial Dyck path characters* [CM15].
- They seem to be *key (Demazure character) positive*.
- Is there a geometric interpretation?
- May have easier transition to other types.

Other avenues

- More cancellation?

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- More specializations?

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- More cancellation?
- More specializations?
- Hanlon's Conjecture:

$$J_\lambda^{(\alpha)}(x) = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \alpha^{f(\sigma,\tau)} \epsilon(\tau) p_{\mathrm{type}(\sigma\tau)}$$

Thank you!

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