Model theory and operator algebras
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Native land aknowledgement

I wish to acknowledge that we are meeting today on the traditional lands of the Ktanaxa, Tsuu T'ina and Niitsitapi peoples.

Correspondences and model theory

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Outline of talk

- A comparision of the use of ultraproducts in model theory and operator algebra
- Ultraproducts as an exploratory tool for finding the right language
- Test case 1: correspondences and property T
- Test case 2: the uniform 2-norm
- Test case 3: σ-finite von Neumann algebras

The role of ultraproducts

Theorem (Łoś' Theorem)

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I$, $\mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{\mathcal{U}} \mathcal{M}_i$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Theorem

Suppose A and M are separable $\mathcal{L}$-structures then TFAE

1. *A embeds into N for some $N \equiv M$.*
2. *A embeds into $M^{\mathcal{U}}$ for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*
3. *A satisifies the universal theory of M.*

Model theoretic use of ultraproducts

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

- *$\mathcal{C}$ is an elementary class i.e. all $\mathcal{L}$-structures satisfying some set of sentences.*
- *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.*
- *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Definable sets via functors

- Met will be the category of bounded metric spaces with isometries as morphisms. Mod(T) is the category of models of T.
- Suppose we have a theory T in a language $\mathcal{L}$ and S_i for $i \leq n$ are sorts in $\mathcal{L}$. We call a functor

$$X \colon \mathrm{Mod}(T) \to \mathrm{Met}$$

  a T-functor if for every model $\mathcal{M}$ of T, $X(\mathcal{M})$ is a closed subset of $\prod_{j=1}^{m} S_j(\mathcal{M})$ and X is just restriction on morphisms.
- This functor is called a *definable set* if for all models $\mathcal{M}$ of T the function $d(x, X(\mathcal{M}))$ is a formula in T.

Definable sets and ultraproducts

Theorem

Suppose that X is a T-functor. Then the following are equivalent:

1. *X is a definable set.*
2. *For all sets I, ultrafilters $\mathcal{U}$ on I and models of T, $\mathcal{M}_i$ for $i \in I$, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then*

$$X(\mathcal{M}) = \prod_{\mathcal{U}} X(\mathcal{M}_i).$$

Background on correspondences

Fix tracial von Neumann algebras (M, τ_M) and (N, τ_N).

- An M-N correspondence is a Hilbert space H together with commuting normal representations π_M and π_N.
- If $\phi : M \to N$ is a completely positive map then on $M \bar{\otimes} N$ define the sesquilinear form

  $$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = \tau_N(\phi(a_2^* a_1) b_2^* b_1).$$

  Let H_ϕ be the correspondence obtained by taking the completion of $M \bar{\otimes} N$ with respect to $\langle \cdot, \cdot \rangle$.
- If H is a correspondence, $\xi \in H$ is K-bounded if for all $c \in M_+$, $d \in N_+$

  $$\langle c\xi, \xi \rangle \leq K\tau_M(c) \text{ and } \langle \xi d, \xi \rangle \leq K\tau_N(d).$$

  If ξ satisfies the first inequality it is called left bounded and if the second, right bounded.

Background, cont'd

- Suppose H is a correspondence, $\xi \in H$ is right bounded and $R_\xi : N \to H$ by the right action. We define $\phi_\xi : M \to N$ by

$$\phi_\xi(m) = R_\xi^* m R_\xi.$$

  $\phi_\xi(m)$ is in fact in N and not just in $B(L^2(N, \tau_N))$ and ϕ_ξ is a c.p. map.
- If ξ is a right bounded vector in a correspondence H then H_{ϕ_ξ} is isomorphic to $\overline{M\xi N}$ via the map which sends $1 \otimes 1$ to ξ.
- Every correspondence is the direct sum of cyclic correspondences of the form H_ϕ where ϕ is a c.p. map associated to a 1-bounded vector or subtracial vector.

Property T for II_1 factors

Definition

We say that a II_1 factor M has property T if for every $\epsilon > 0$ there is a finite $F \subseteq M$ and $\delta > 0$ such that if H is an M-M correspondence, $\xi \in H$ is a unit vector and $\|[x, \xi]\| \leq \delta$ for all $x \in F$ then there is a central vector $\eta \in H$ such that $\|\eta - \xi\| \leq \epsilon$.

- Problem 1: We don't have a model theory of correspondences.
- Problem 2: We don't have a notion of ultraproduct for correspondences.

The language of correspondences

Fix tracial von Neumann algebras M and N. The language $\mathcal{L}$ of M-N correspondences will include:

- for each $K \in \mathbb{N}$, there will be a sort S_K and for any correspondence H, $S_K(H)$ will be the set of K-bounded vectors. The metric will be induced by the inner product on H;
- for $K < L$ there will be an isometry from S_K to S_L which for a given correspondence will be interpreted as the inclusion map;
- $+$ will be defined on all pairs of sorts and will be interpreted standardly as the restriction of addition from any correspondence; and,
- there will be unary functions for each $c \in M$ and $d \in N$ which implement the left and right actions.

The equivalence

- Let $\mathrm{Corr}(M, N)$ be the category of M-N correspondences.
- For $H \in \mathrm{Corr}(M, N)$, let $\overline{H}$ be the $\mathcal{L}$-structure described on the previous slide, called the dissection of H, and $\mathcal{C}$ be the class of all such structures.
- We want to show two things:
  1. $\mathcal{C}$ is an elementary class, and
  2. the functor $H \to \overline{H}$ is an equivalence.

Ultraproducts of correspondences

1. Fix M-N correspondences H_i for $i \in I$ and an ultrafilter $\mathcal{U}$ on I. We can form the ultraproduct in two ways:
2. We could take the ultraproducts of the dissections. This amounts to forming $S_K(H_i)$ for each K and i and let H, the ultraproduct, be the closure of

$$\bigcup_K \left(\prod_{\mathcal{U}} S_K(H_i) \right)$$

   in $\prod_{\mathcal{U}} H_i$.
3. Alternatively, we could take those $\xi \in \prod_{\mathcal{U}} H_i$ at which the left and right actions at ξ are continuous i.e. $L_\xi : M \to \prod_{\mathcal{U}} H_i$ and $R_\xi : N \to \prod_{\mathcal{U}} H_i$ are bounded.

The main theorem

Theorem

1. *If M and N are tracial von Neumann algebras then the class of M-N correspondences forms an elementary class.*
2. *If M is a II_1 factor then M has property T iff the set of 1-bounded M-central vectors is a definable set for the class of M-M correspondences.*
3. *The class of M-N correspondences is model theoretically very nice: it is stable, classifiable and has a model companion.*

Uniform 2-norm

- Fix a C^*-algebra A and a non-empty set Φ of states on A. For $x \in A$, define

$$\|x\|_\Phi = \sup_{\varphi \in \Phi} \|x\|_\varphi .$$

- This is a semi-norm on A and we say Φ is faithful if $\|\cdot\|_\Phi$ is a norm.
- Already $(A, \|\cdot\|, \|\cdot\|_\Phi)$ is a metric structure but we want one more condition.
- We say Φ is full if it is invariant under unitary conjugation. We now assume Φ is full and faithful.

Uniform 2-norm, cont'd

Now mimic the construction of the standard representation: let $L^2(A, \Phi)$ be the Banach space completion of A with respect to $\|\cdot\|_\Phi$. We call $\xi \in L^2(A, \Phi)$ K-bounded if

$$\|a\xi\|_\Phi \leq K\|a\|_\Phi \text{ and } \|\xi a\|_\Phi \leq K\|a\|_\Phi.$$

Let A_Φ be the Banach algebra of all bounded vectors in $L^2(A, \Phi)$. There is a natural involution on A_Φ arising from the adjoint on A and this makes A_Φ an involutive Banach algebra. Let's call A_Φ the statial algebra associated to Φ.

Proposition

If Φ is full and faithful then A_Φ admits an equivalent C^*-algebra norm.

Uniform 2-norm and ultraproducts

- For each $i \in I$, fix $(A_i, \|\cdot\|, \|\cdot\|_{\Phi_i})$ for C^*-algebras A and full and faithful Φ_i, and let $\mathcal{A}_i$ be its associated statial algebra. Let $\mathcal{U}$ be an ultrafilter on I. Here are three equivalent ways to view the ultraproduct of the $\mathcal{A}_i$'s:
- Form $\prod_{\mathcal{U}} A_i$ as a C^*-algebra and consider its left and right actions on the Banach space ultraproduct $\prod_{\mathcal{U}} L^2(A_i, \Phi_i)$. Let $\prod_{\mathcal{U}} \mathcal{A}_i$ be the closure of the points of continuity of these actions.
- Equivalently, the $\prod_{\mathcal{U}} \mathcal{A}_i$ could be the closure of the bounded points of this action.
- A third possibility is that we could take the ultraproduct of the metric structures $(A_i, \|\cdot\|, \|\cdot\|_{\Phi_i})$ and then let $\prod_{\mathcal{U}} \mathcal{A}_i$ be the closure of the bounded points of $L^2(\prod_{\mathcal{U}} A_i, \|\cdot\|_{\hat{\Phi}})$ where $\hat{\Phi} = \lim_{\mathcal{U}} \Phi_i$.
- The last takes advantage of the fact that the statial algebra is an imaginary sort.

Observations

Theorem (Ozawa,Ng-Robert,Rørdam,BBSTWW)

Suppose A is a simple, exact, $\mathcal{Z}$-stable C^*-algebra in which all quasi-traces are traces. Then if $\Phi = T(A)$, $\|\cdot\|_\Phi$ is equivalent to a $\mathcal{L}_{\mathrm{C}^*}$-formula in the theory of A.

Suppose that A is a C^*-algebra and Φ is full and faithful. Then we have a short exact sequence

$$0 \to J \to A^{\mathcal{U}} \to A^{\mathcal{U}}_\Phi \to 0$$

where J is the kernel of the quotient map. By demanding that Φ is faithful, the map from A to A_Φ is injective. This implies that the kernel of the quotient map is not a definable set.

σ-finite von Neumann algebras

- A von Neumann algebra M is σ-finite if it has a faithful, normal state. We let σ-vNa be the class of all pairs (M, φ) where M is a vNa and φ is a faithful normal state on M.
- For $(M, \varphi) \in \sigma$-vNa, let H_φ be the standard representation arising from M via φ. M acts naturally on H_φ on the left.
- We say that $a \in M$ is φ-right K-bounded if

$$\langle ba, ba \rangle_\varphi \leq K \langle b, b \rangle \text{ for all } b \in M.$$

- Fact: If (M, φ) is a σ-finite vNa then the set of φ-right bounded elements of M is strongly dense.

The Ocneanu ultraproduct

- Fix σ-finite vNa's (M_i, φ_i) for $i \in I$ and $\mathcal{U}$, an ultrafilter on I.
- For a state φ on a σ-finite vNa, write $\|x\|_{\varphi}^{\#}$ for

$$\sqrt{\varphi(x^*x) + \varphi(xx^*)} \text{ and let}$$

$$\ell^{\infty}(M_i, I) = \{(a_i) \in \prod_I M_i : \sup_I \|a_i\| < \infty\} \text{ and}$$

$$J = \{(a_i) \in \ell^{\infty}(M_i, I) : \lim_{\mathcal{U}} \|a_i\|_{\varphi_i}^{\#} = 0\}.$$

- Then the Ocneanu ultraproduct is $M(J)/J$ with faithful normal state given by $\lim_{\mathcal{U}} \varphi_i$.

σ-finite von Neumann algebras - the language

Fix a σ-finite (M, φ).

- We will have sorts $S_{K,N}(M)$ for the set of all $x \in M$ such that $\|x\| \leq N$ and both x and x^* are φ-right K-bounded. The norm on these sorts is $\|\cdot\|_\varphi^\#$. The sorts are complete with respect to this norm.
- This leads to a natural notion of the dissection of (M, φ) where we restrict all the algebraic operations to the sorts.
- Let $\mathcal{S}$ be the class of all such dissections.

σ-finite von Neumann algebras - the theorem

Theorem

1. *(Dabrowski) The class σ-vNa is categorically equivalent to an elementary class.*
2. *The class $\mathcal{S}$ is an elementary class which is categorically equivalent to σ-vNa. Write T_σ for the theory of $\mathcal{S}$.*

If (M, φ) is a σ-finite let Δ_φ be the modular operator with respect to φ and let

$$\sigma_t(x) = \Delta^{it} x \Delta^{-it} \text{ for } t \in \mathbb{R}.$$

Corollary

For each $t \in \mathbb{R}$, σ_t restricted to any sort S is T_σ-definable. If this restriction is given by a formula ψ_t^S then the map $t \mapsto \psi_t^S$ is continuous in the logic topology.

Thank you!