Universality in a Large Ensemble of F-theory Geometries

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Based on work with Jim Halverson and Benjamin Sung

What is F-theory?

- Generalization of the typical framework of string theory; allows for more general physics than is realizable in weakly-coupled string theory.

- Dictionary between a Calabi-Yau elliptic (or genus-one) fibration **X** over a base **B**, and a physical theory. **Morrison, Vafa**

$$X \to B \quad \sim \text{physical theory}$$

- Here **B** is the geometry of physical extra spatial dimensions. The fibration encodes data necessary to determine gauge group, matter content, Yukawa coupling etc. that are all important data for a physical theory.

- Elliptically-fibered Calabi-Yau fourfold **X** with threefold base **B** is most relevant for physics of our universe, since this gives a physical theory in four dimensions. We will focus on such fourfolds.

Physics questions

- Different elliptically fibered **X** define different physical theories, and we are therefore immediately led to asking some questions:

1. Is the set of **X** finite in any sense? **See talk by Di Cerbo, Svaldi**
2. If so, how many are there? What are their features?
3. Are models that give physics close to that of our universe common or uncommon? Is there a geometric reason for this?
4. Are there any universal features that may serve as a sort of "prediction"?

A great deal of progress: many people here!

- This talk will concentrate on (2) and a little bit (4), which I will elaborate on.

A few details

- Elliptic fibration parametrized by Weierstrass equation

$$y^2 = x^3 + f(z_i)x + g(z_i)$$

f and **g** are global sections of line bundle:

$$f \in \Gamma(\mathcal{O}(-4K_B)) \qquad g \in \Gamma(\mathcal{O}(-6K_B))$$

- Singularities of the elliptic fibration play an important role. The subloci where the elliptic fiber becomes singular is given by the discriminant locus:

$$\Delta = 4f^3 + 27g^2 = 0$$

- Physically, the vanishing of the discriminant marks the location of 7-branes, which are divisors in **B**. Let **z** be a local coordinate where the fiber becomes singular. The MOV (a, b, c) along **z= 0** of (**f, g,** Δ), respectively, determine, by Kodaira's classification of singular fibers, a geometric gauge group, which is some of the data for our physical theory.

$$f = z^a F \qquad g = z^b G \qquad \Delta = z^c \tilde{\Delta}$$

Non-Higgsable 7-branes

F	a	b	c	Sing.	G
I_0	≥ 0	≥ 0	0	none	none
I_n	0	0	$n \geq 2$	A_{n-1}	$SU(n)$ or $Sp(\lfloor n/2 \rfloor)$
II	≥ 1	1	2	none	none
III	1	≥ 2	3	A_1	$SU(2)$
IV	≥ 2	2	4	A_2	$SU(3)$ or $SU(2)$
I_0^*	≥ 2	≥ 3	6	D_4	$SO(8)$ or $SO(7)$ or G_2
I_n^*	2	3	$n \geq 7$	D_{n-2}	$SO(2n-4)$ or $SO(2n-5)$
IV^*	≥ 3	4	8	E_6	E_6 or F_4
III^*	3	≥ 5	9	E_7	E_7
II^*	≥ 4	5	10	E_8	E_8

Kodaira

- In some cases, we may have **a > 0** and **b > 0** for all choices of complex structure moduli.

Morrison, Taylor

- The locus **z = 0** then has a <u>non-Higgsable 7-brane</u>. We say these are non-Higgsable because the fiber type is independent of complex structure, and the gauge group therefore cannot be broken/changed by a complex structure deformation (Higgsing).

Some selective progress: Halverson, Grassi, Morrison, Shaneson, Taylor, Wang

Bases for F-theory

- What bases give well-defined F-theory models? Not every base will do. First,

$$\mathcal{O}(-4K_B) \quad \mathcal{O}(-6K_B)$$

must have global sections to construct the elliptically fibered Calabi-Yau.

- Second, we need the base to give sensible physics. We have good control over the theory when we can make the Calabi-Yau **X** <u>smooth</u>. We can directly control the model when (not mutually exclusive):

1. **X** admits a crepant resolution.

2. **X** admits a deformation in complex structure to a smooth Calabi-Yau.

- However, we don't expect that the only physically admissible **B** are those for which **X** is smoothable; in particular, if **X** is at finite distance in moduli space from another elliptically fibered Calabi-Yau **X'** , we expect **X'** to be a physically reasonable as well.

Base transitions

- Starting with an elliptically fibered Calabi-Yau **X -> B**, one can crepantly pass to another elliptically fibered Calabi-Yau **X" -> B'** by a base-change, and pass to a minimal Weierstrass model.

- This procedure is

  1. Perform a blowup **B' -> B** in the base along a subvariety **C** and perform a base change
  $$X^{'} = X \times_B B^{'} \to B^{'}$$
  2. Perform a change of coordinates and pass to a minimal Weierstrass model **X" -> B'**.

Candelas, Diaconescu, Florea, Morrison, Rajesh

- For this procedure to be crepant we need

$$MOV_C(f,g) \geq (4,6)$$ **if C is a curve in B**

$$MOV_C(f,g) \geq (8,12)$$ **if C is a point in B**

- This produces a new elliptic Calabi-Yau **X" -> B'**, with a new base **B'** which is a blowup of **B**.

Finite distance in moduli space

- On the other hand, we'd like to be at finite distance in moduli space from a Calabi-Yau we understand.

- For elliptically fibered Calabi-Yau fourfolds (with smooth threefold base), **X"** is finite distance from **X** if

$$MOV_D(f,g) < (4,6)$$ **for all divisors** D

$$MOV_C(f,g) < (8,12)$$ **for all curves** C

$$MOV_P(f,g) < (12,18)$$ **for all points** P

Hayakawa, Wang

Candelas, Diaconescu, Florea, Grassi, Morrison, Rajesh

We'll refer to this generally as the (4,6) condition

Strategy for generating bases

- Our goal will be to generate a large ensemble of bases, and understand the associated physics.

- This is an explorative approach to understanding bases for F-theory: see what we can generate and what we can learn from it.

- The strategy will be to:

  1. Start with some "minimal" geometry whose associated physics we understand (i.e. we can smooth it with complex structure deformation, or there is a crepant resolution).

  2. Move through Calabi-Yau moduli space by performing base transitions.

  3. Do so without violating the (4,6) condition, so that each Calabi-Yau is connected in moduli space to the original model.

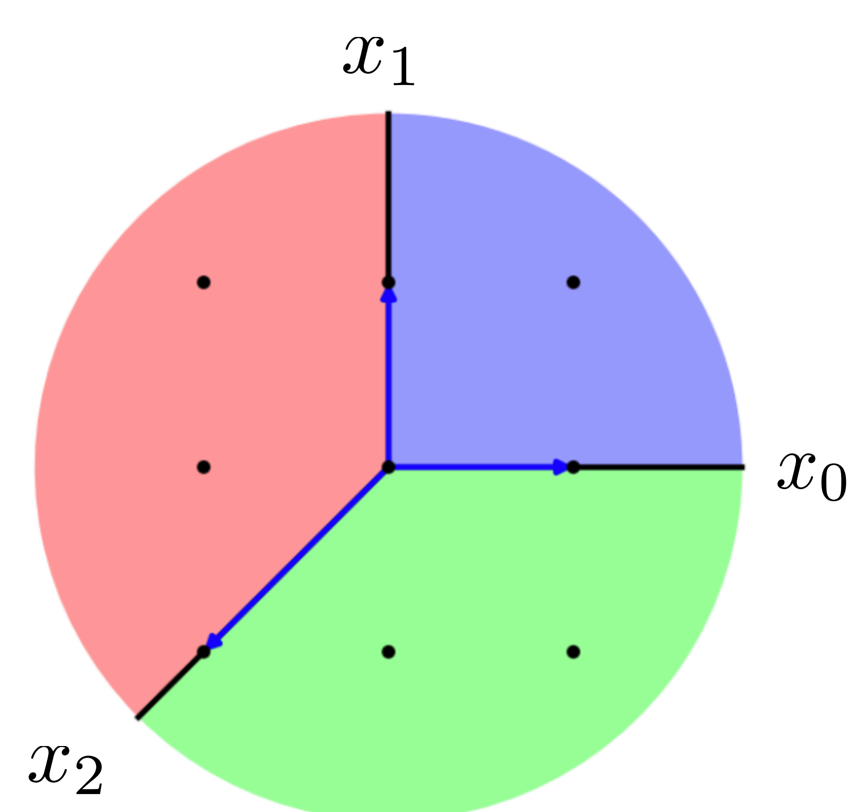
- We will use <u>toric threefolds</u> as bases, which are combinatorial, and very nice to work with.

Toric combinatorics

- Toric varieties are combinatorial: they admit a description in terms of a fan of rational polyhedral cones.
- Each ray of the fan v_i corresponds to a homogeneous (toric) coordinate x_i and therefore each ray corresponds to a divisor

$$D_i = \{x_i = 0\}$$

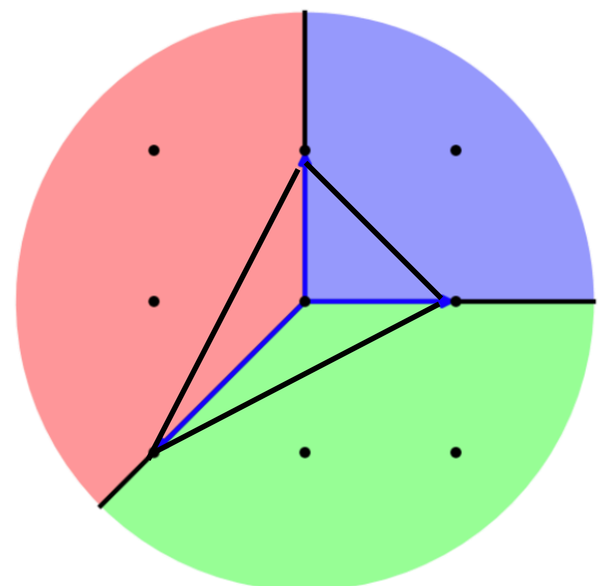
- n-dimensional cones then correspond to codimension-n subvarieties, by setting the corresponding toric coordinates to zero.
- Example: $\mathbb{P}^2$



Toric varieties and polytopes

- Some fans correspond to face fans of triangulations of boundaries of reflexive polytopes. Such reflexive polytopes give a rich class of toric varieties.

Batyrev, Kreuzer, Skarke



A fine, regular, star triangulation (FRST) of a 3d reflexive polytope corresponds to a smooth projective toric 3-fold.

- These toric varieties are weak Fano toric varieties (WFTV), and the generic CY 4-fold elliptic fibrations over them are smooth, which implies there are no non-Higgsable 7-branes, and no gauge groups generically.

- There are 4319 3d reflexive polytopes, and ~10^15 triangulations total, and so these are a rich class of toric threefolds.

Halverson, Tian, Carifio, Kriokov, Nelson

Weak Fano toric threefolds as minimal geometries

- As stated before, WFTVs have no non-Higgsable 7-branes. Can tune gauge groups by tuning degenerating fibers, but are not forced with any.

- We will generate the ensemble via toric blowups of these WFTVs.

- Blowups can only decrease the number of sections in **f** and **g**, which increases the likelihood of having a NH7, and moves us closer to the (4,6) condition.

- For a given toric variety V, corresponding to a fan F with rays v_i and homogenous coordinates x_i, a global section of $\mathcal{O}(-nK_B)$ corresponds to

$$\{m | \langle m, v_i \rangle \geq -n \, \forall \, v_i\}$$

- The corresponding section is then of the form

$$\prod_i x_i^{\langle m, v_i \rangle + n}$$

Blowups along toric subvarieties

- Consider a WFT threefold given by a FRST **T** of a 3d reflexive polytope. Toric subvarieties are toric points and toric curves. Toric points are represented by 2-simplices (faces) in **T**, and toric curves are represented by 1-simplices (edges) in **T**.

- Blowups are done by adding rays. For a subvariety corresponding to the cone

$$\{v_1, \ldots, v_n\}$$

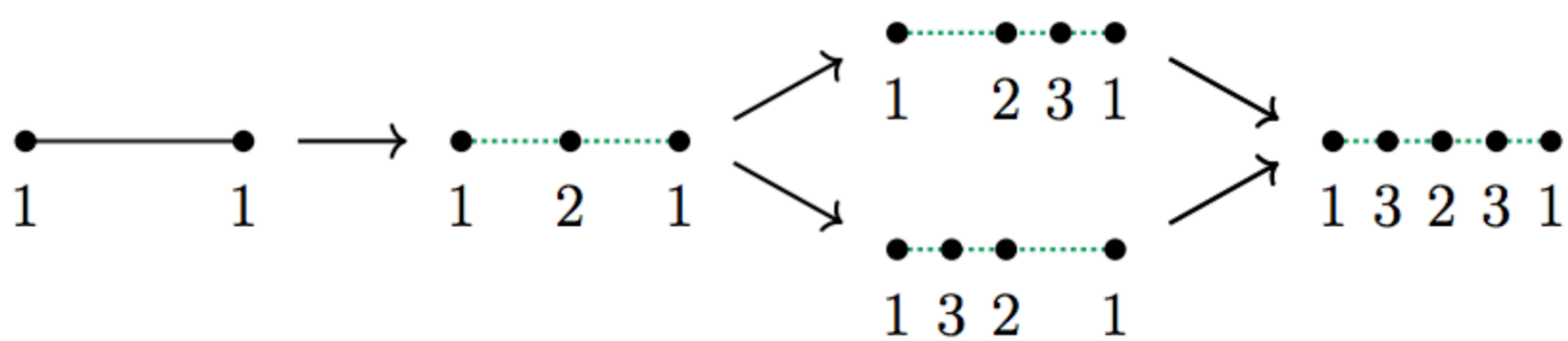
one blows up this subvariety by adding a ray

$$v_e = \sum_i v_i$$

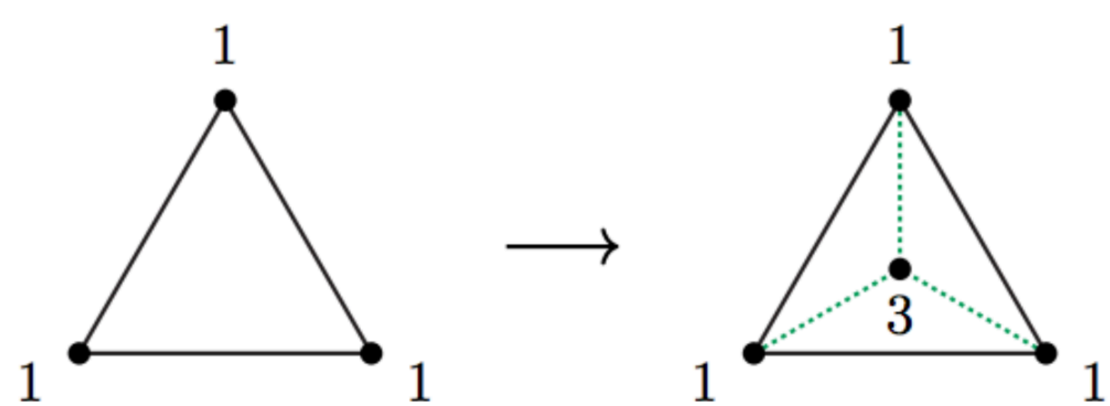
Can iterate this to continually blowup and form a chain of new varieties from the original one.

Blowups along toric subvarieties

- For visualization it is easiest to project down the exceptional rays arising from the blowups down to the polytope, so blowing up corresponds to subdividing simplices



Blowing up a toric curve



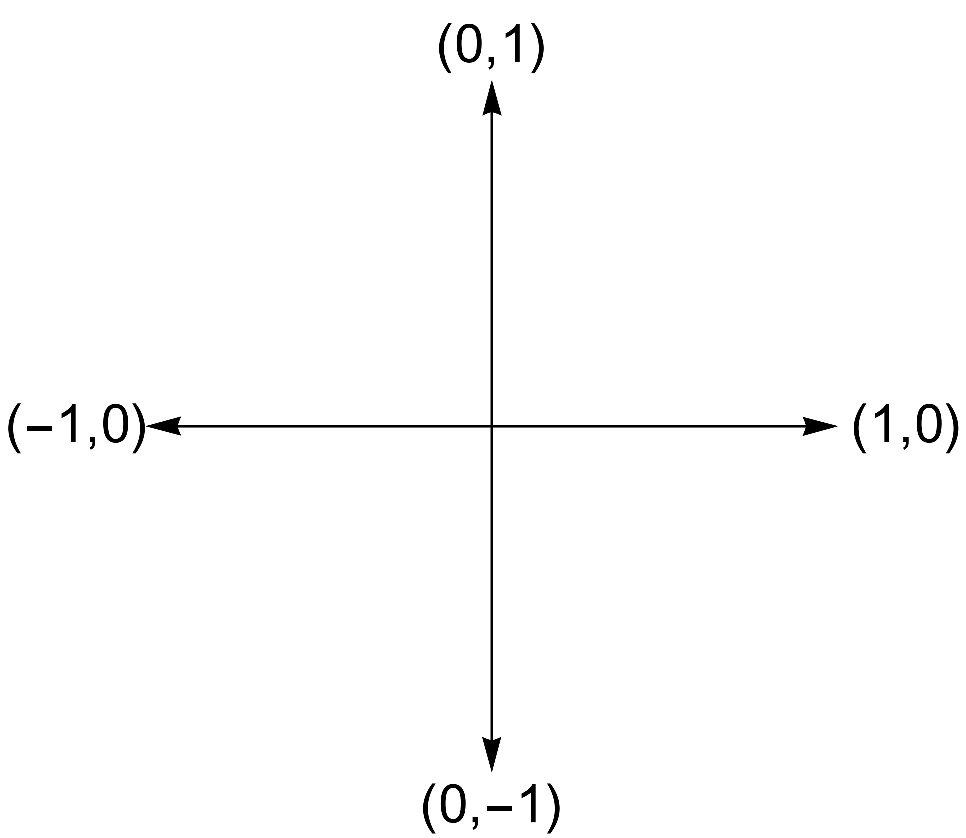
Blowing up a toric point

Blowups cut out sections

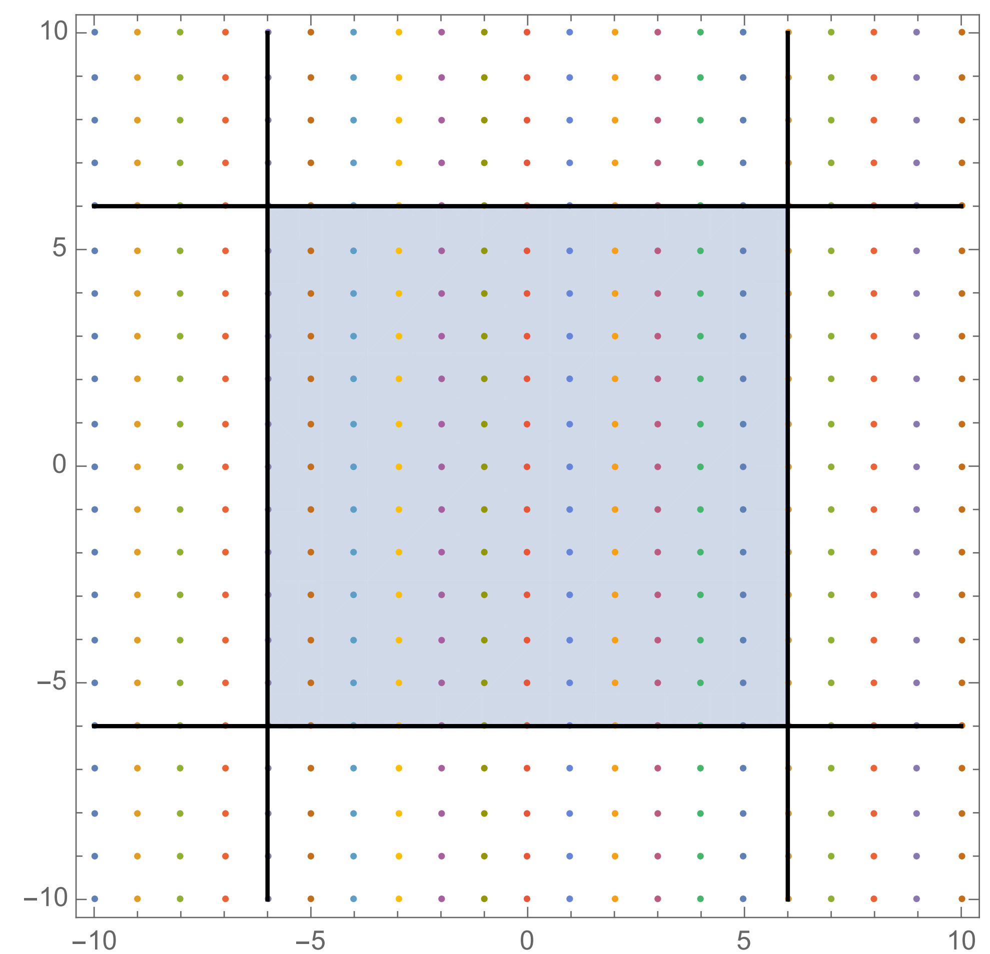
- Adding additional rays to the fan adds additional hyperplane constraints, and generically decreases the number of global sections of

$$\mathcal{O}(-nK_B), n \geq 0$$

- Example: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}(-4K_B)$:

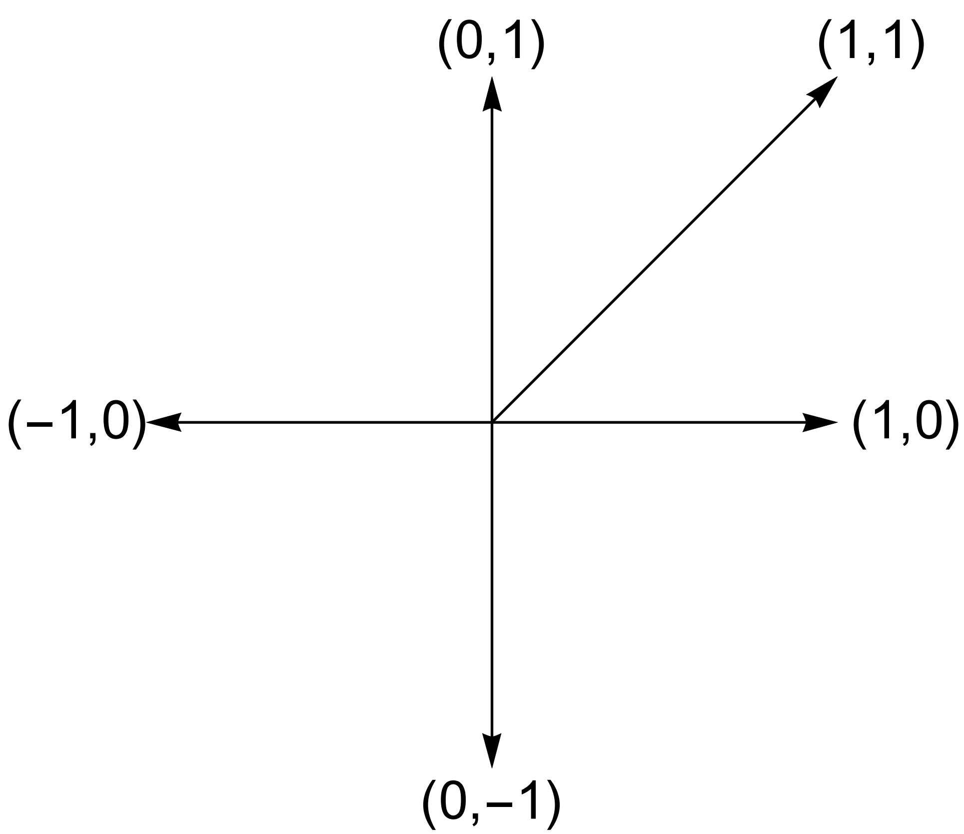


Fan



Global Sections

Example: $\mathbb{P}^1 \times \mathbb{P}^1$



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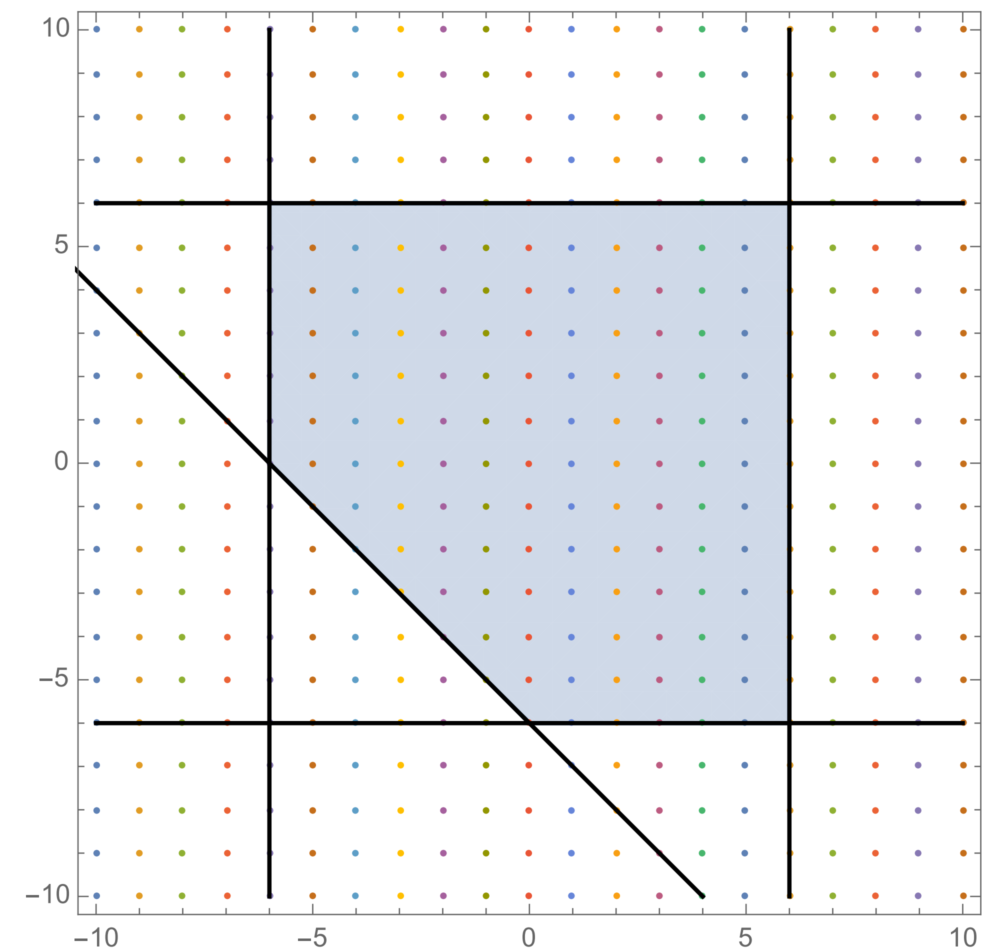

Global Sections

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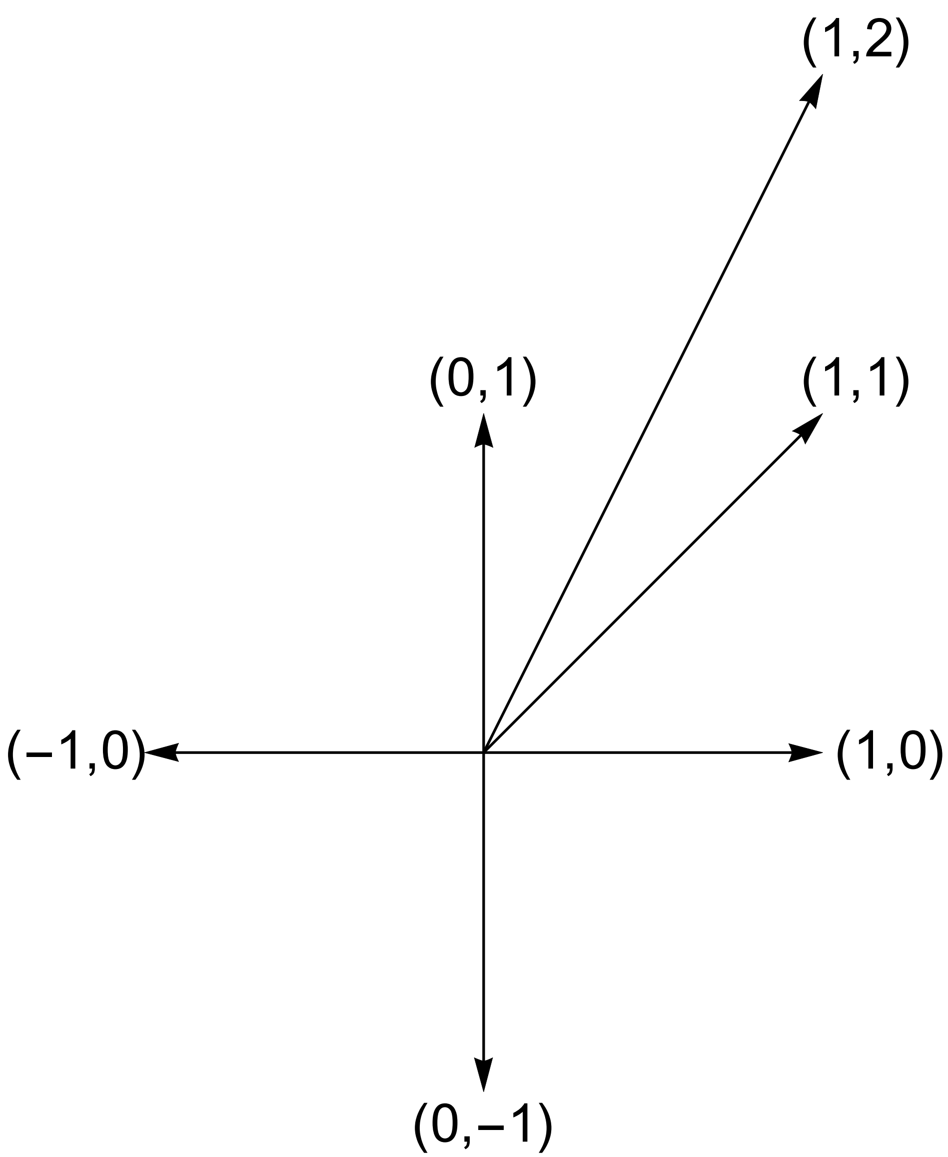

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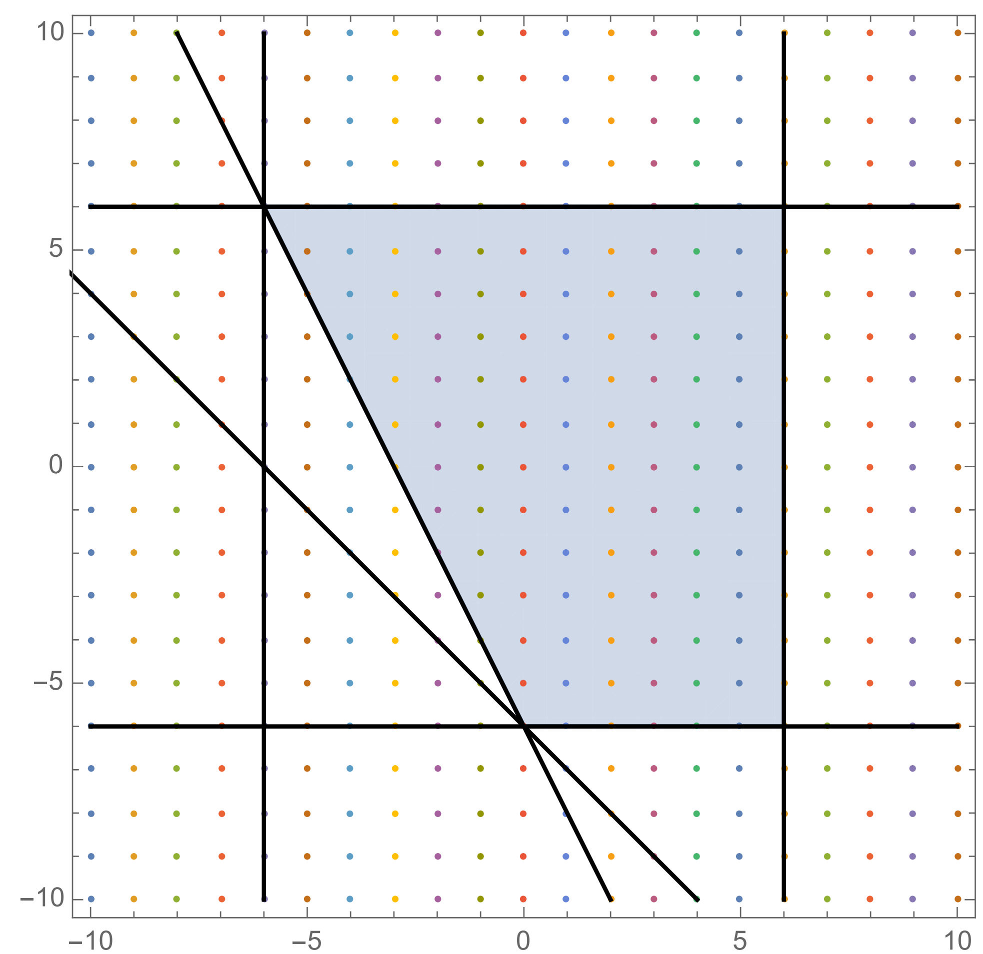

Global Sections

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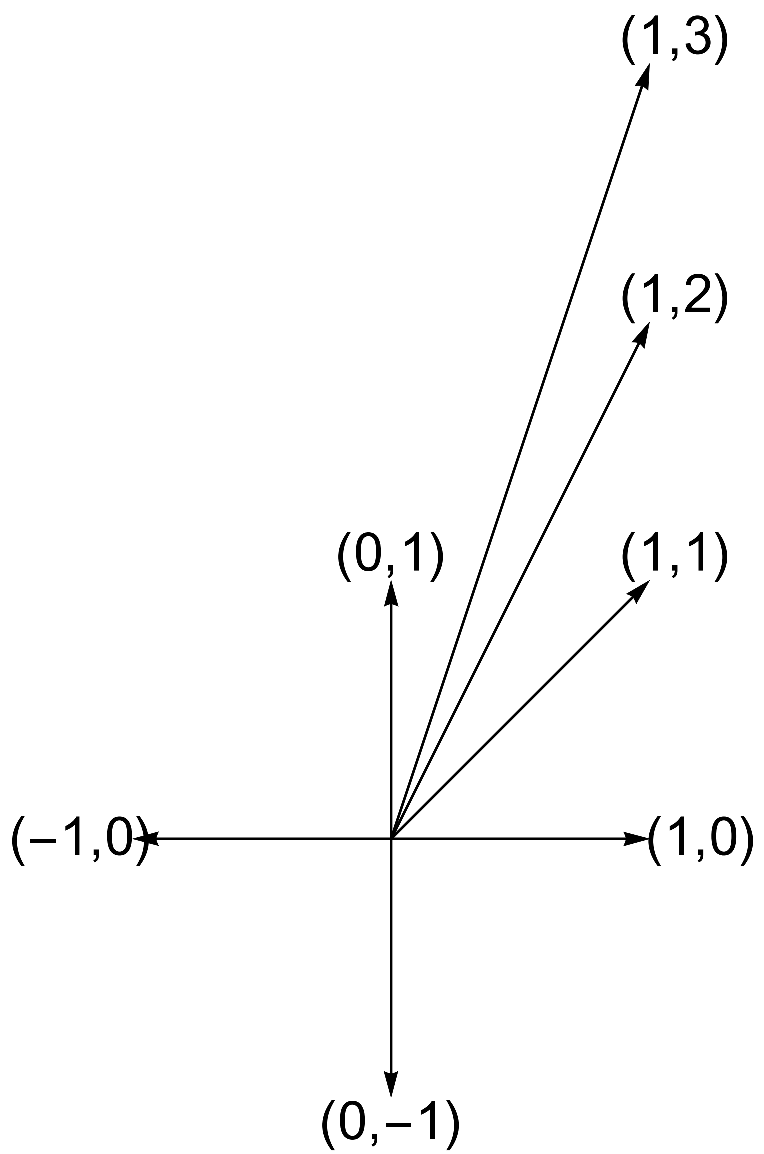

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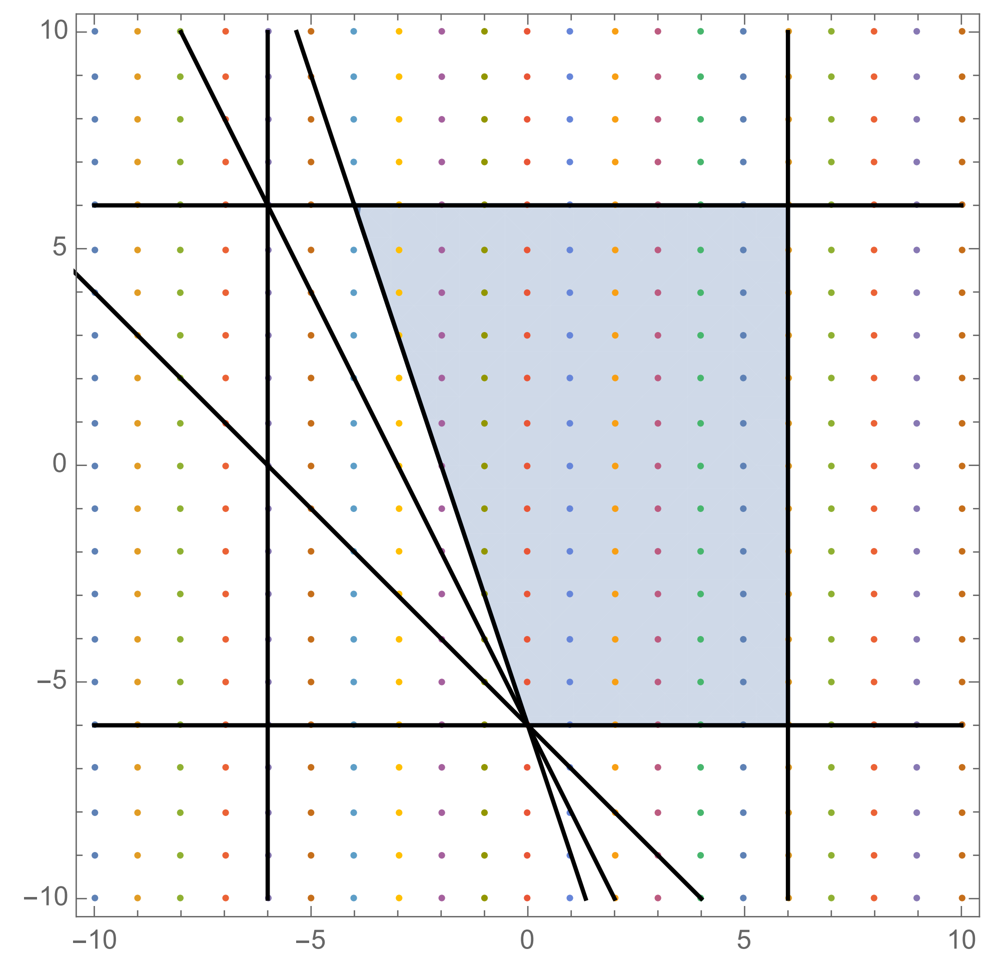

Global Sections

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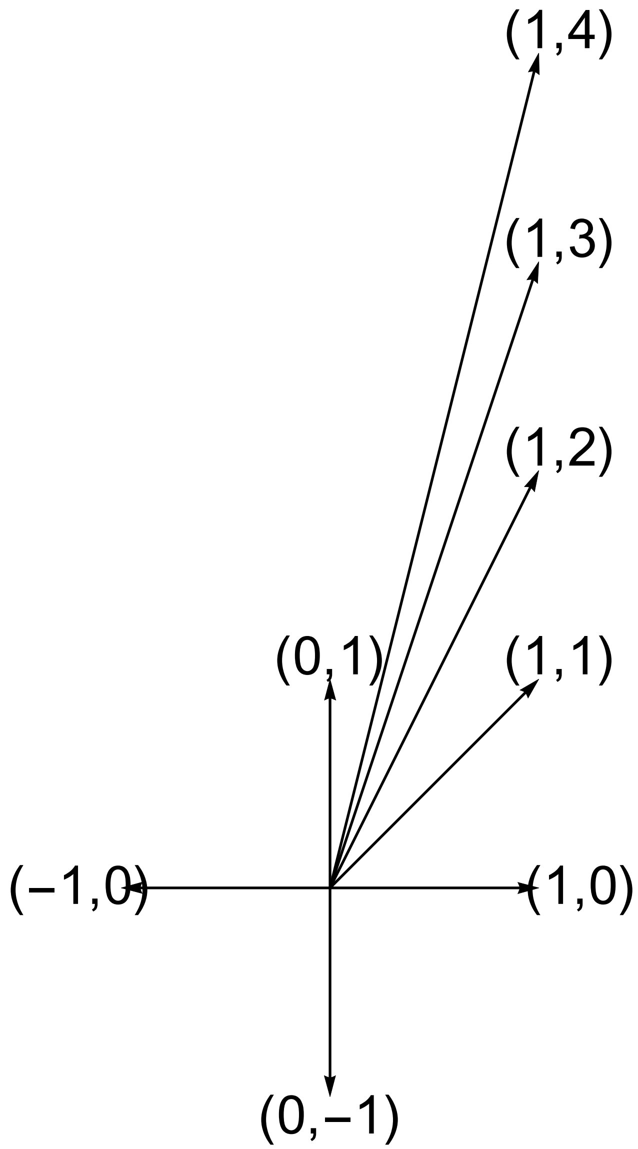

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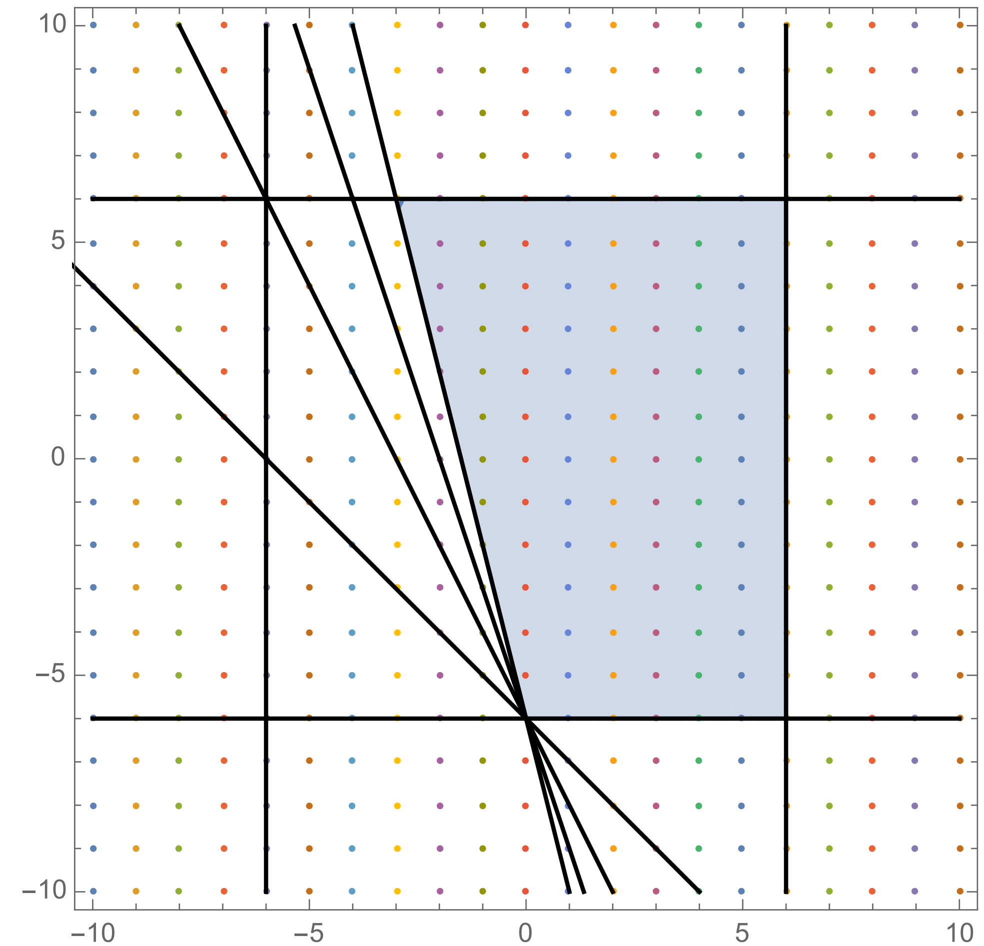


Global Sections

- Intuitively, these blowups force higher MOV, and move us from smooth elliptic fibration towards ones that saturate the (4,6) condition. We'll see this is true in more detail later.

Strategy again

- Now we have a refined strategy:

    1. Start with a WFT threefold base, and corresponding elliptically fibered Calabi-Yau fourfold.

    2. String together as many toric blowups as possible to produce new bases and corresponding Calabi-Yau fourfolds, without violating the (4,6) condition.

- The ensemble of such geometries turns out to be enormous (we don't know the actual size).

- Two complimentary approaches to understand/probe this space:

    1. Sample the space using random walks, use statistical techniques to estimate the size and other properties of the ensemble (See Yinan's talk).

       Taylor, Wang

    2. Find a way to <u>factorize</u> (some of) the blowups in order to derive a strict lower bound on the number of admissible bases (this talk).

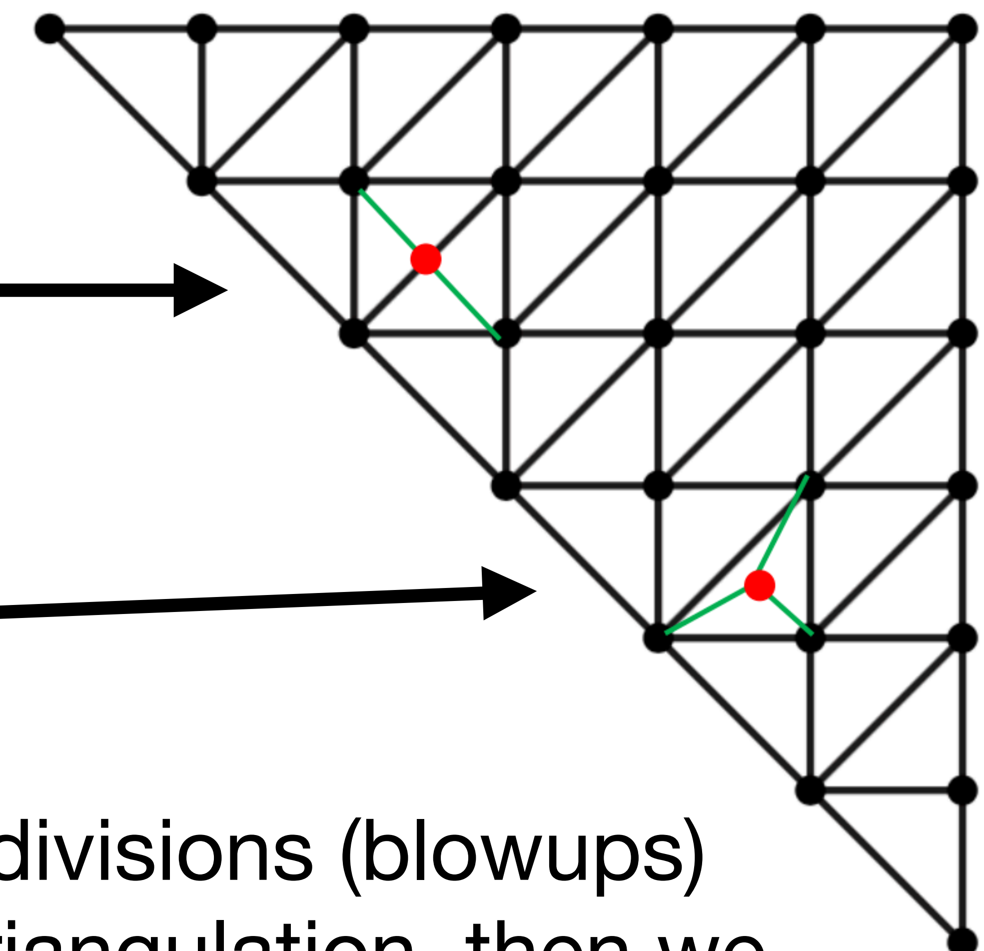
       Halverson, C.L., Sung

Factorizing blowups

- Easiest to describe in combinatorial language with a picture. Consider this FRST of a face of a 3d reflexive polytope, and the following observation about toric blowups:

Subdivision of edges does not respect original FRST of polytope.

Subdivision internal to a face (2-simplex) DOES respect original FRST of polytope!



- If we first consider sequences of subdivisions (blowups) internal to each face on the original triangulation, then we can perform such blowups without affecting the toric fan elsewhere, and so can work locally with each face. We can then subdivide each edge, corresponding to blowups of toric curves.

The height bound

- We can now consider possible subdivisions of faces and edges. A priori there is an infinite number of such subdivisions.

- However, we need to fulfill the (4,6) condition, which will prevent us from performing arbitrary blowups.

- Recall that the ray in the toric fan corresponding to an exceptional divisors from a blowup can be written as

$$e = \sum_i a_i v_i$$

where the v_i are points on the original reflexive polytope.

- We define the <u>height</u> as $$h = \sum_i a_i$$

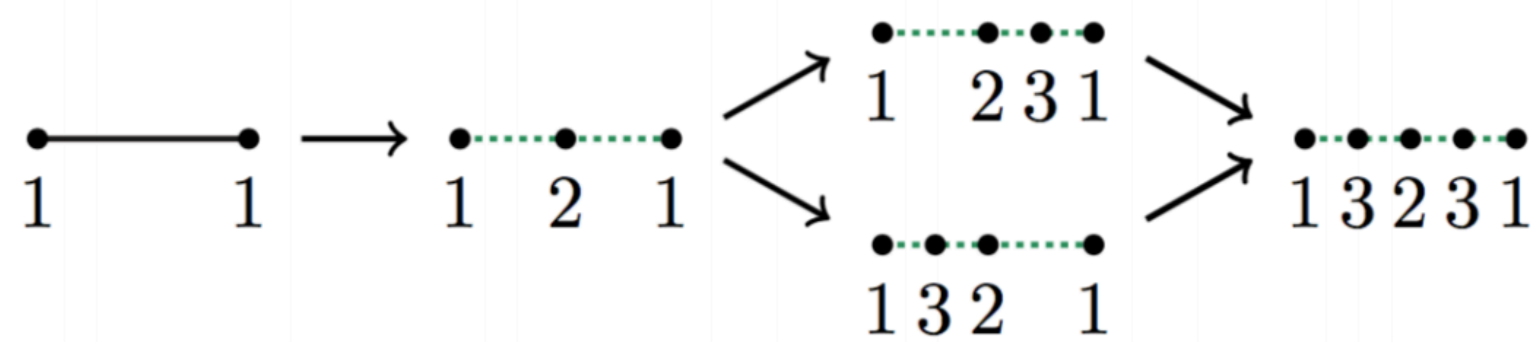
- **Theorem: if h ≤ 6 for all rays in the fan corresponding to the toric threefold base, then the (4,6) condition is satisfied.**

Trees of geometries

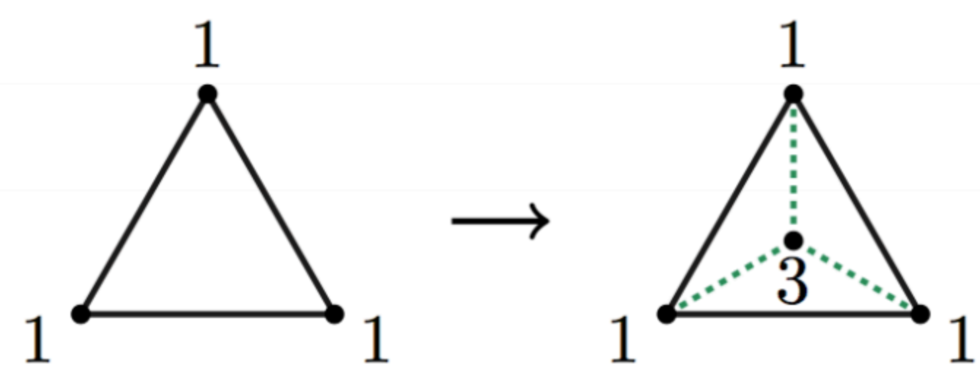
- We’ll refer to a sequence of blowups as a “tree”, exceptional ray in fan from blowup as a “leaf”
- Trees over edges = “edge trees”
- Trees over faces = “face trees”
- Points on polytope = “roots”
- Need to classify all trees with h ≤ 6 for all leaves.
- Do so by exhaustively constructing the toric blowups.

Classification of trees

- All 5 $h \leq 3$ edge trees.



- Both $h \leq 3$ face trees.



- # for $h \leq N$:

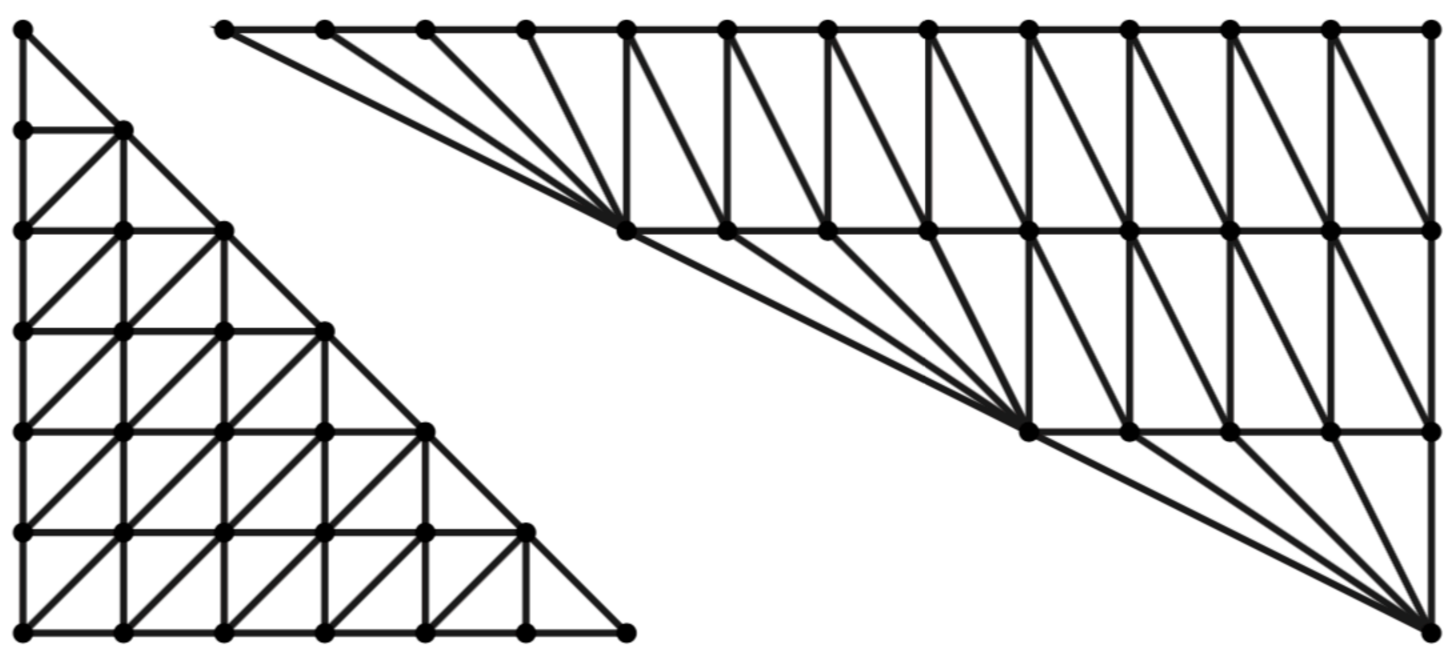
N	# Edge Trees	# Face Trees
3	5	2
4	10	17
5	50	4231
6	82	41,873,645

Forests from trees

- Now that we've classified all of the trees (local subdivisions), we can start with a FRST of a reflexive polytope, and construct an ensemble of geometries by building trees over it to perform base transitions (a forest).

- Given the number of face-trees, the reflexive polytope whose FRST has the largest number of faces will dominate the ensemble (independent of particular triangulation).

- There are two such polytopes, each with 108 edges and 72 faces. Each has a very large facet:



large face of Δ_1° **large face of** Δ_2°

Ensemble of compact bases

- Trees over these two large polytopes generate

$$\frac{2.96}{3} \times 10^{755} \quad , \quad 2.96 \times 10^{755}$$

bases, respectively. Each one is explicitly constructable!

- All other reflexive polytopes give $\leq 3.28 \times 10^{692}$

- This is an explicit lower bound on the number of admissible geometries for F-theory, not an estimate.

Physics things we learn from these bases

- Now we have some large ensembles. Do these large ensembles display any universal physical behavior? What can we learn from these ensembles?

- Look for high-probability features with interesting associated physics.

   1. Non-Higgsable 7-branes are universal.

   2. The gauge group has generically high rank.

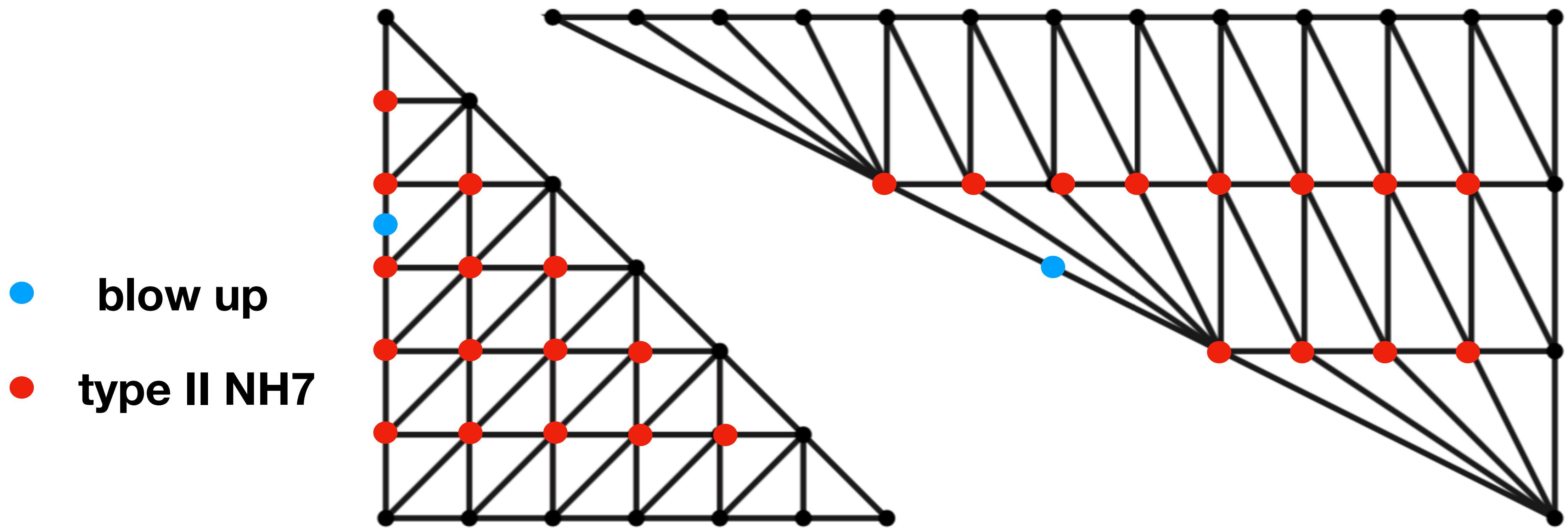
   3. Weak coupling limits, which connect F-theory to weakly-coupled string theory, do not exist generically.

- These conclusions follow from facts about reflexive polytopes and some linear algebra, so I will just report results.

1. Universality of NH7

- Consider an edge or facet of a polytope, and perform a height > 2 blowup on that edge or facet.

- This cuts out a special monomial in **f**, **g**, forces type II NH7 on all divisors corresponding to points interior to the edge or facet.



All face trees (except for one on ground) have a h > 2 leaf.

All but two edge trees have a h > 2 leaf.

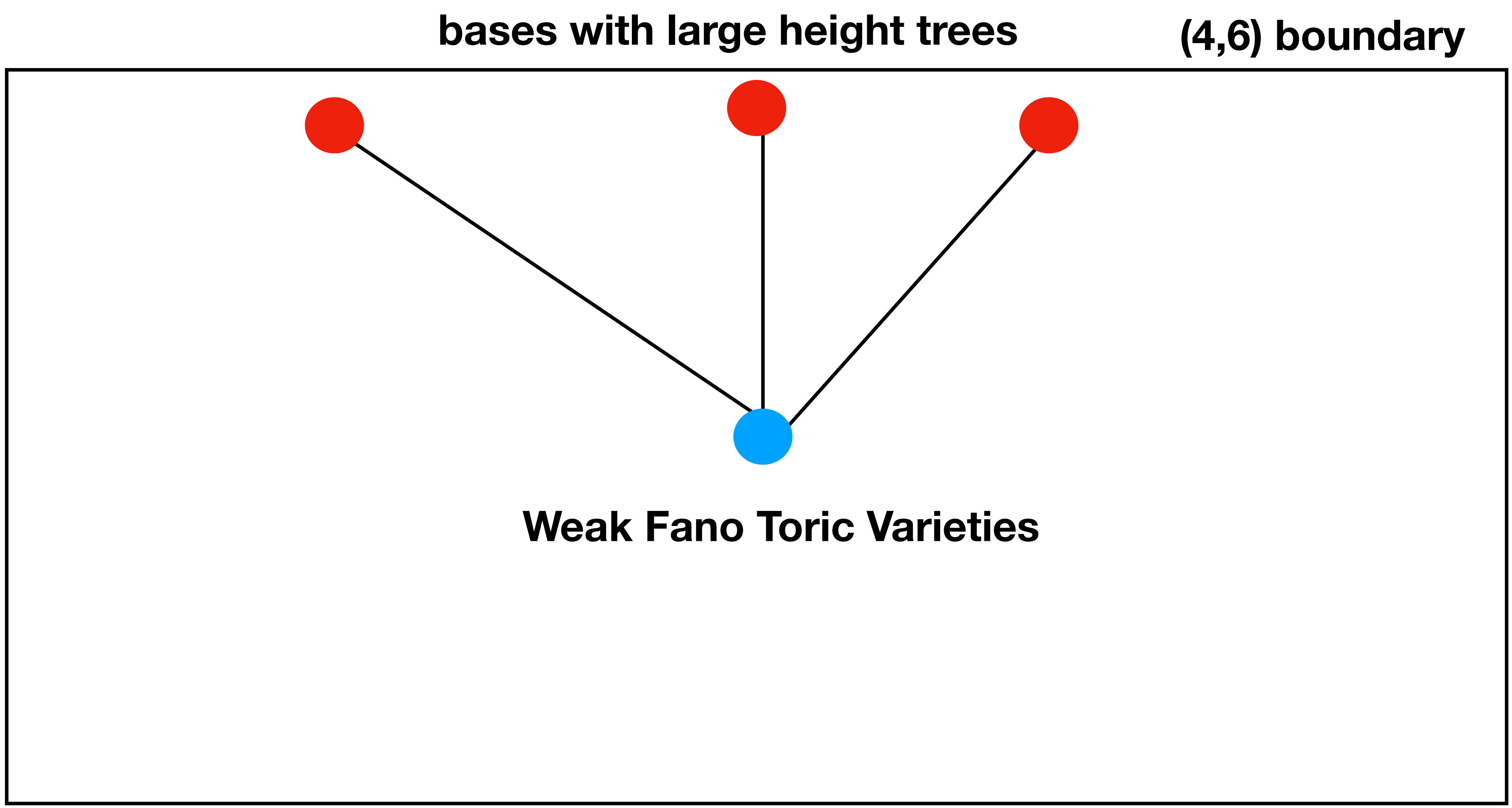
$$P(\text{NHC in } S_{\Delta_1^\circ}) \geq 1 - 1.01 \times 10^{-755}$$

$$P(\text{NHC in } S_{\Delta_2^\circ}) \geq 1 - .338 \times 10^{-755}$$

Aside: Weak Fanos in the space of bases

- This behavior is universal: higher height blowups force NH7 with higher and higher vanishing in **f** and **g**, push us further towards (4,6) boundary (but not all the way when h <= 6).

- Gives the following cartoon for the bases we consider:



2. Algorithmic universality of gauge groups

- E_8 roots are very common; require only a few blowups on the large facets to force E_8 on all points interior to the facets.

- Thm: A leaf built on g roots with height h = 1,2,3,4,5,6 has Kodaira fiber $F = II^*, IV^*_{ns}, I^*_{0ns}, IV_{ns}, II, -$ and geometric gauge group $E_8, G_2, SU(2), -, -$ respectively.

- Let H_i be number of height i leaves above E_8 roots.

- We then have a gauge group of the form

$$G \geq E_8^{10} \times F_4^{18} \times U^9 \times F_4^{H_2} \times G_2^{H_3} \times A_1^{H_4} \qquad U \in \{G_2, F_4, E_6\}$$

$$rk(G) \geq 160 + 4H_2 + 2H_3 + H_4$$

with probability $\geq .999995$

- Confirmed by machine learning.

3. No weak coupling limits

- A particular limit in complex structure moduli space, known as the Sen's limit, can take us from F-theory to a weakly coupled type IIB limit. Sen

- Sen's limit requires us to tune only smooth, I_n, and I_n^* fibers, on the geometry $(n \geq 0)$.

- This is generically obstructed due to NH7 on rigid divisors with too large $MOV_D(f, g)$ (higher than I_0^*).

3. No weak coupling limits

A Sen limit is spoiled by:

- A height-2 blowup along two 1-simplices on the same edge, coupled with a height-3 blowup along that same edge.
- A height-3 blowup along a 2-simplex strictly interior to a face.

- Fraction of geometries that admit a Sen limit: $< 3 \times 10^{-391}$

Halverson, C.L., Sung

- These geometries are inherently strongly-coupled F-theory geometries that do not admit a weakly coupled string theory description.

Beyond toric bases

- Can we generalize beyond toric bases? Observation: the minimal geometries we've considered (WFTV) can be viewed as patching together crepant resolutions of orbifold singularities of $\mathbb{C}^3$ of the form:

  Roan

  1. Isolated singularities. **Degeratu, Yau**

  2. A_n singularities fibered over curves.

- A natural generalization to move beyond toric threefolds is to consider crepant resolutions of other orbifold singularities. What's left are

  1. D_n singularities fibered over a curve.

  Joyce

  2. E_n singularities fibered over a curve.

  Facchini, Gonzalez-Alonso, Lason

- By looking at the Cox ring of the resolutions, we find that building any trees above these geometries forces non-Higgsable clusters on rigid divisors arising in the crepant resolution, and spoils the existence of a Sen's limit. These geometries produce inherently strongly coupled physics as well!

Halverson, C.L., Sung

4. Aside: network/graph structure

- Interesting feature: these geometries have a graph structure, where the edges in the graph are the topological transitions. **Taylor, Wang**

- Of course, the graph is not a tree.

- This network structure may play an interesting physical role: there are physical processes that allow a universe in one vacuum/geometry to nucleate other universes in other vacua/geometry via bubble nucleation, gives rise to a rich cosmological history.
  Coleman, De Lucia, Garriga, Schwartz-Perlov, Vilenkin, Winitzki,

- If the graph structure plays a role in transitions (i.e. determining minimal actions instantons between geometries) it may give rise to vacuum selection determined by the geometry via a model of bubble cosmology on the network. **Carifio, Cunninnham, Halverson, Krioukov, C.L., Nelson**

- A simple model of bubble cosmology on our network picks out a geometry with geometric gauge group:

$$E_8^{37} \times F_4^{85} \times G_2^{220} \times SU(2)^{320}$$

Summary and thoughts

- Base transitions from a geometry we understand can generate large ensembles of admissible F-theory geometries.

- In the approach we took (factorizing blowups), we set a lower bound of

$$\frac{4}{3} \times 2.96 \times 10^{755}$$

  geometries for F-theory.

- By understanding the construction algorithm we were able to understand universal results about the ensemble without actually generating it.

- These geometries exhibit non-Higgsable clusters, have generically high rank gauge groups, and for the most part do not admit weak coupling limits.

Summary and thoughts

- It would be very interesting to understanding if such universality arguments can be made in other ensembles, i.e. the one generated by random walks by Taylor and Wang.

- Questions: To what extent do toric bases capture features of general F-theory bases?

- How can one add flux to these geometries, when they do not admit a full crepant resolution or a deformation to smooth Calabi-Yau? What is the distribution of flux vacua?

- How do we understand instantons interpolating between different singular geometries?

Thanks!

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