# Multiple Fibrations, in Calabi-Yau Geometries

James Gray – Virginia Tech

Based on work with:

Alexander Haupt and Andre Lukas:
arXiv:1303.1832, arXiv:1405.2073

Lara Anderson, Xin Gao and Seung-Joo Lee
arXiv:1608.07554, 1608.07555 & 1708.07907

Lara Anderson and Brian Hammack
arXiv:1803.XXXXX

# Complete Intersection Calabi-Yau (CICYs)

- A family of CICYs is described by a configuration matrix:

$$[\mathbf{n}|\mathbf{q}] \equiv \left[\begin{array}{c|ccc} n_1 & q_1^1 & \ldots & q_K^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q_1^m & \ldots & q_K^m \end{array}\right]$$

with m rows and K+1 columns.

- Ambient space is $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$
- Remaining columns give degree of defining relations:

Calabi-Yau condition:

$$\sum_{\alpha=1}^{K} q_\alpha^r = n_r + 1$$

D-fold condition:

$$\sum_r n_r - K \stackrel{!}{=} D$$

# Example:

- An example of a configuration matrix (CICY four-fold 244):

$$\left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^3 & 0 & 4 \end{array}\right]$$

- The different choices of defining relation corresponds to a redundant description of part of complex structure moduli space:

$$p_1 = \sum_{i,a} c_{i,a} x^i y^a \quad p_2 = \sum_{i,\ldots,\delta} d_{iab\alpha\beta\gamma\delta} x^i y^a y^b z^\alpha z^\beta z^\gamma z^\delta$$

- This example is a Calabi-Yau four-fold.

# CICY Data Sets:

- Three-Folds:

  - Hübsch, Commun.Math.Phys. 108 (1987) 291
  - Green et al, Commun.Math.Phys. 109 (1987) 99
  - Candelas et al, Nucl.Phys. B 298 (1988) 493
  - Candelas et al, Nucl.Phys. B 306 (1988) 113

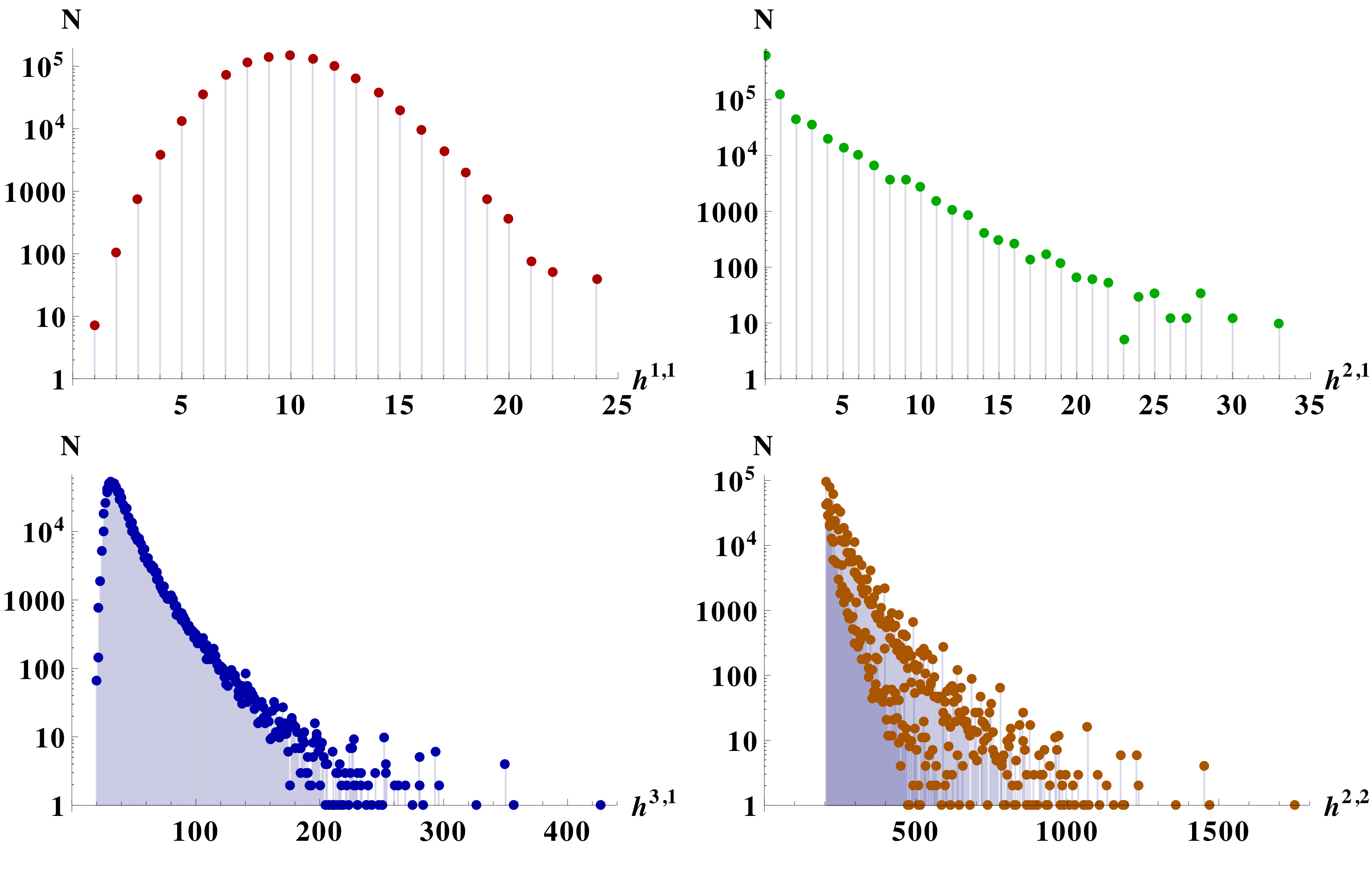
- Data Set classified: 7890 configuration matrices in the set.
- This data set has been used extensively in the study of compactifications of heterotic string theory.

- Four-Folds:

    - Brunner et al, Nucl.Phys. B498 (1997) 156-174
    - JG et al, JHEP 1307 (2013) 070
    - JG et al, JHEP 1409 (2014) 093

- Data set classified: 921,497 configuration matrices in the set.
- Technology is being developed to use this data set for studying F-theory compactifications– as I will describe later.
- All Hodge data etc. are available for these manifolds:

# Example: fourfold Hodge data



# Properties of CICYs: Torus Fibrations

- Consider configuration matrices which can be put in the form:

$$\left[\ \mathcal{A}_1\ \middle|\ \mathcal{F}\ \right] = T^2$$

$$\left[\begin{array}{c|cc} \mathcal{A}_1 & 0 & \mathcal{F} \\ \mathcal{A}_2 & \mathcal{B} & \mathcal{T} \end{array}\right]$$

Base: $\left[\ \mathcal{A}_2\ \middle|\ \mathcal{B}\ \right]$

- This is an torus fibred four-fold
- Essentially all CICYs are fibered in this manner. For example 7837 out of 7890 threefolds (99.3%)

- Example:

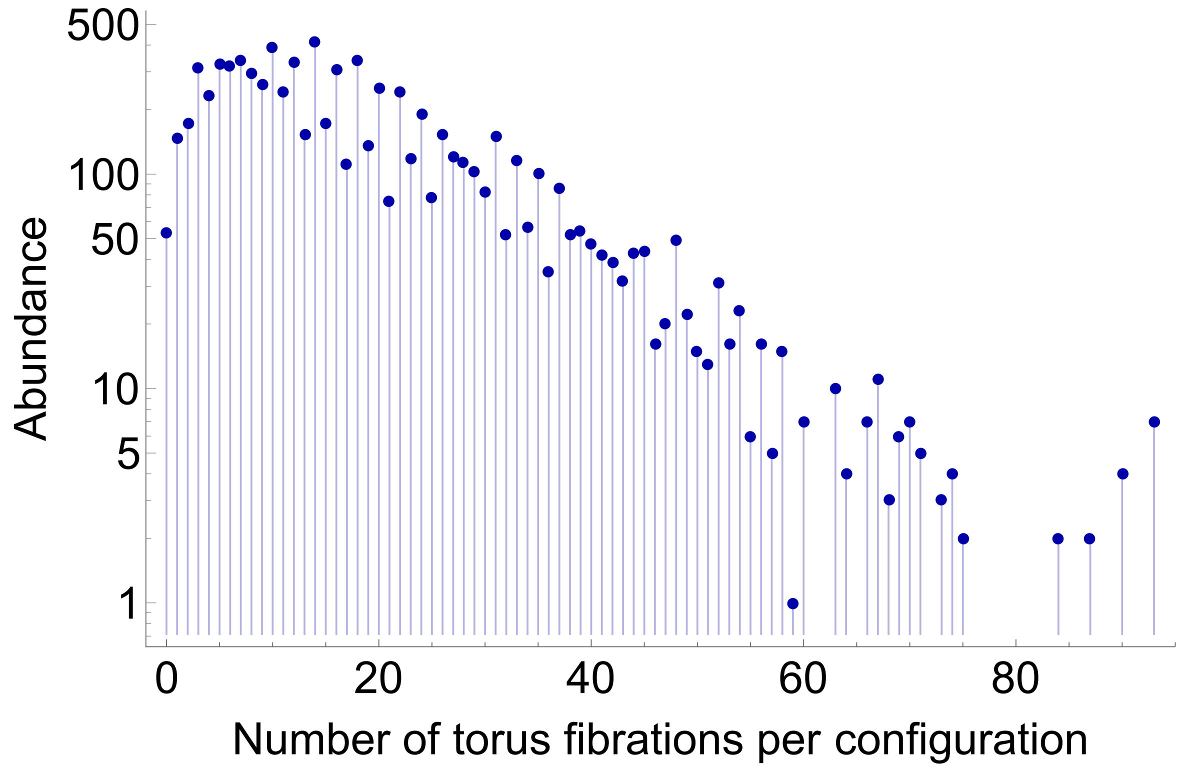
$$
\left(\begin{array}{c|cc:cccc}
\mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\
\mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hdashline
\mathbb{P}^1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\mathbb{P}^1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\mathbb{P}^2 & 1 & 2 & 0 & 0 & 0 & 0
\end{array}\right)
$$

- This is not an artifact of the threefolds. For fourfolds 921,020 out of 921,497 configuration matrices are obviously torus fibered in this way (99.9%)
- See also related work for other constructions: arXiv:1406.0514 and 1605.08052 by S. Johnson and W. Taylor.
- A given manifold/configuration matrix may admit many obvious elliptic fibrations…

# Number of torus fibrations per threefold:

- 139,597 fibrations in total.

- The average CICY threefold admits 17.7 different fibrations




- The largest number of fibrations admitted by one example is 93.

- In our simple example we also have:

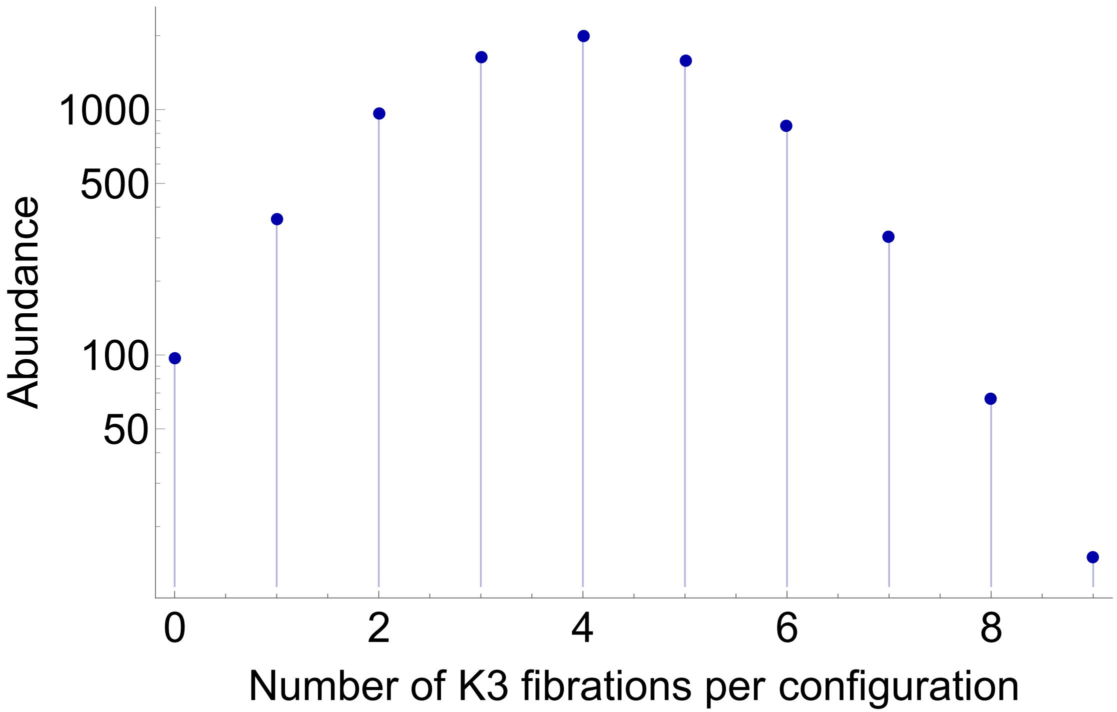
$$\left(\begin{array}{c|c|ccccc} \mathbb{P}^1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^2 & 2 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{c|c|ccccc} \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c||cccccc} \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \left(\begin{array}{c||cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c||cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \end{array}\right)$$

- Note that we have a variety of different bases here (Hirzebruchs, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ etc in this case).
- It doesn’t just have to be *torus* fibration structures that exist in a CICY…

# Number of K3 fibrations per threefold:

- 98.5% of CICY threefolds are K3 fibered.
- 30,974 fibrations in total
- The average CICY threefold admits 3.9 different fibrations




- The largest number of fibrations admitted by one example is 9.

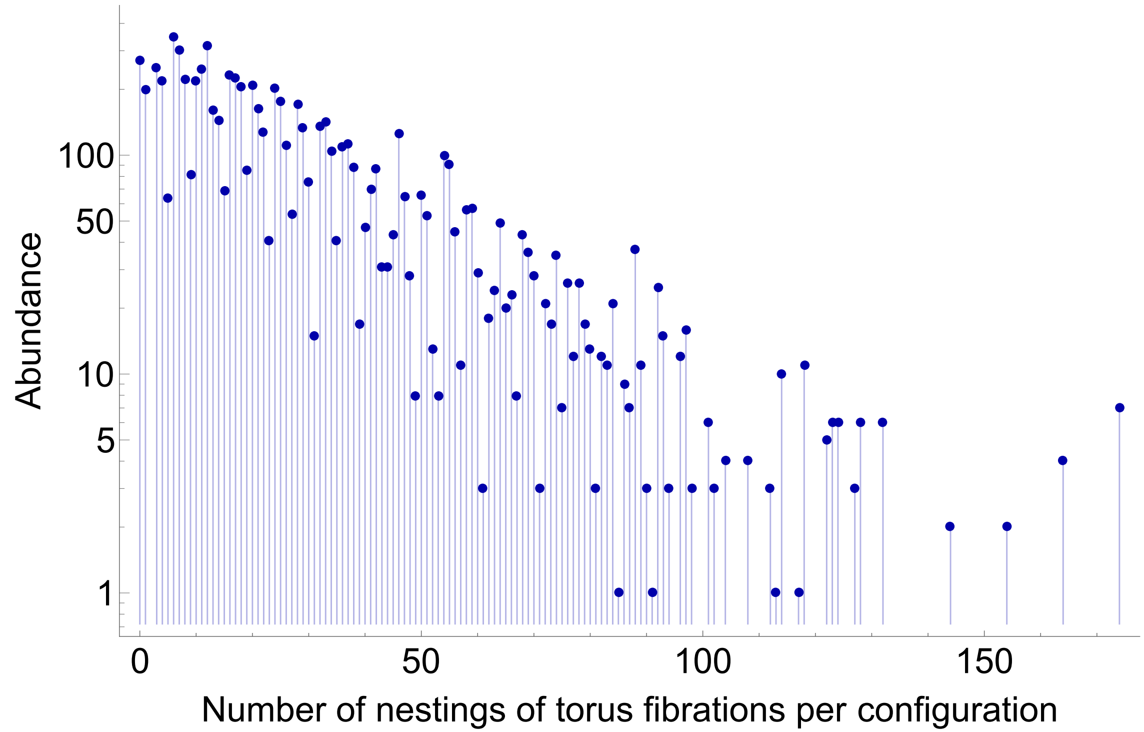
- In our simple example:

$$\left(\begin{array}{c||cccccc} \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ \hdashline \mathbb{P}^1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{c||cccccc} \mathbb{P}^2 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^1 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

- Again this example is slightly less rich than the average case…
- One could ask if the K3 fibers are elliptically fibred…

# Number of nested fibrations per threefold:

- 208,987 torus fibrations nested in K3 fibrations.
- The average CICY threefold admits 26.6 different such structures.




- The largest number of such nested fibration structures admitted by one example is 174.

- Note these numbers are bigger than the related numbers for torus fibrations on their own…
- Example in our case:
(there are six in total in the two K3 fibrations)

$$\left(\begin{array}{c|cc|cccc}
\mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\
\mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hdashline
\mathbb{P}^1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\mathbb{P}^2 & 1 & 2 & 0 & 0 & 0 & 0 \\
\hdashline\hdashline
\mathbb{P}^1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right)$$

# Can we go beyond these obvious fibrations?

- Conjecture by Kollar (rough description):

  *A Calabi-Yau threefold is genus one fibered if and only if there exists a divisor* $D$ *such that*

$$D \cdot C \geq 0 \quad \textit{for every algebraic curve} \quad C$$

$$D^3 = 0$$

$$D^2 \neq 0$$

*(and similarly in higher dimensional cases)*

- *Proven in threefold case by Oguiso, Wilson.*

- The question is, do we have good computational control over all of the elements of $h^{1,1}$?
- In favorable cases we do. For example in the case,

$$X = \left[\begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array}\right]$$

all divisor classes descend from divisor classes in the ambient space.

- In non-favorable cases we don't. For example

$$X' = \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 3 & 0 \\ \mathbb{P}^2 & 0 & 3 \end{array}\right]$$

has $h^{1,1} = 19$ but $h^{1,1}$ of the ambient space is only $3$.

- Of 7890 CICY threefolds in the original list, only 4874 are favorable.

- We can obtain new configuration matrices describing the same manifolds by the process of contraction/splitting:

$$\left[\begin{array}{c|ccccc} n & 1 & 1 & \ldots & 1 & 0 \\ \mathbf{n} & \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_{n+1} & \mathbf{q} \end{array}\right] \longleftrightarrow \left[\begin{array}{c|cc} \mathbf{n} & \sum_{a=1}^{n+1} \mathbf{u}_a & \mathbf{q} \end{array}\right]$$

Euler number doesn't change $\Leftrightarrow$ manifolds same

  - Use this to increase the size of the ambient space affording the configuration a better chance of being favorable

- By splitting we have obtained favorable descriptions of all but 7842 of the 7890 CICYS.

- We can then compute data such as intersection numbers, line bundle cohomology etc completely in these cases.

# What about the remaining 48?

- It turns out that these can all be written as hypersurfaces in direct products of del Pezzo surfaces.
- For example:

$$X_3 = \left[ \begin{array}{c|cccc} \mathbb{P}^1 & 1 & 0 & 0 & 1 \\ \mathbb{P}^2 & 2 & 0 & 0 & 1 \\ \mathbb{P}^4 & 0 & 2 & 2 & 1 \end{array} \right]$$

  can be written as the anti-canonical hypersurface inside

$$dP_4 = \left[ \begin{array}{c|c} \mathbb{P}^1 & 1 \\ \mathbb{P}^2 & 2 \end{array} \right] \quad \text{times} \quad dP_5 = \left[ \begin{array}{c|cc} \mathbb{P}^4 & 2 & 2 \end{array} \right]$$

- Enough is known about the divisors of del Pezzo's that we can then find a favorable description of these spaces too.

Thus we find a favorable description of all CICYs.

- The final ingredient required to investigation the fibrations of CICYs is knowledge of the Kahler cone.
- We have been able to show that the Kahler cone descends simply from the ambient product of projective spaces in 4874 cases (we call these Kahler favorable).
- For the Kahler favorable cases, obvious fibrations and Kollar fibrations coincide.

  However, in general there can be many more Kollar fibrations than obvious ones.
- A good example is the Split-Bicubic/Schoen manifold – which admits an infinite number of genus one fibrations!

(See also Grassi,Morrison; Aspinwall, Gross; Oguiso; Piateckii-Shapiro, Shafarevich).

# Fibrations and quotients

- One can create a new (non-simply connected) Calabi-Yau by quotienting a CICY by a freely acting symmetry.
- Example: Take the bicubic:

$$X = \left[ \begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right]$$

- With homogeneous coordinates:

$$x_{a,i} \qquad\qquad a = 1, 2 \quad i = 0, 1, 2$$

- And quotient by the following $\mathbb{Z}_3$ group action:

$$g : x_{a,j} \to \omega^j x_{a,j}$$

- Clear in this case, the quotienting preserves the fibration.

- More generally what can we say about fibrations in quotients of CICYs?
- Classification of symmetries:
    - Braun, JHEP 1104 (2011) 005

*(The equivalent classifications for the four-folds has not yet been carried out.)*

- A lot of work has already been done classifying the properties of the associated quotients:
    - Candelas et al, arXiv:1602.06303
    - Braun et al, arXiv:1512.08367
    - Candelas et al, arXiv:1511.01103
    - Constantin et al, arXiv:1607.01830

Unpublished work with Lara Anderson and Brian Hammack:

- Of the 1632 symmetry-CICY pairs (for manifolds with fibration), 1552 of them preserve *some* fibration (95%).
- Of 20700 fibration/symmetry pairs, 17161 preserved.

| Symmetry | Fibs preserved | Fibs not preserved | %preserved |
|---|---|---|---|
| $\mathbb{Z}_2$ | 8812 | 464 | 95% |
| $\mathbb{Z}_3$ | 175 | 201 | 46.5% |
| $\mathbb{Z}_4$ | 120 | 244 | 33.0% |
| $\mathbb{Z}_5$ | 0 | 30 | 0.0% |
| $\mathbb{Z}_6$ | 62 | 438 | 12.4% |
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | 7711 | 1488 | 83.8% |
| $\mathbb{Z}_2 \times \mathbb{Z}_4$ | 105 | 200 | 34.4% |
| $\mathbb{Z}_3 \times \mathbb{Z}_3$ | 176 | 0 | 100% |

- There are several larger symmetries that appear (including non-Abelian symmetries), none of which preserve any fibrations:

$$\mathbb{Z}_8 \ , \ \mathbb{Z}_{10} \ , \ \mathbb{Z}_{12} \ , \ Q_8 \ , \ \mathbb{Z}_2 \times Q_8 \ , \ \mathbb{Z}_3 \rtimes \mathbb{Z}_4 \ ,$$
$$\mathbb{Z}_8 \times \mathbb{Z}_2 \ , \ \mathbb{Z}_4 \rtimes \mathbb{Z}_4 \ , \ \mathbb{Z}_8 \rtimes \mathbb{Z}_2 \ , \ \mathbb{Z}_4 \times \mathbb{Z}_4 \ ,$$
$$\mathbb{Z}_{10} \times \mathbb{Z}_2$$

- In any case where the fibration is preserved, the base of the quotiented fibration is divided by same group as total space.
- Classifications of the bases that appear will be provided in the paper.

# Multiple fibrations and F-theory

- We can use these multiple nested fibration structures to derive some interesting dualities in F-theory. For example:
    - Start with two different fibrations of the same Calabi-Yau. This will correspond to two F-theory models that share an M-theory limit.
    - Start with two different fibrations of the same Calabi-Yau in a heterotic compactification. These will have seemingly different F-theory duals which actually give the same physics.
    - And so on…

# Example:

- Let us consider the first of those possibilities in this case:

$$\left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 \\ \mathbb{P}^2 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^2 & 1 & 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{c|c:ccc} \mathbb{P}^2 & 0 & 1 & 2 & 0 \\ \mathbb{P}^2 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 1 & 1 \end{array}\right]$$

  Just considering these two possible fibrations – one with $\mathbb{P}^2$ and one with $\mathbb{F}_1$ base.

- To analyze this it would be nice to put these two cases in Weierstrass form (blow down every component of the fiber that doesn't intersect zero section).

- To do this we need sections of these fibrations.

- Necessary conditions that a divisor, $\mathcal{S}$, describing a section must obey:
  - Oguiso (intersection number with fiber should be generically one).
  - A condition on the cohomology of the associated line bundle:

$$h^0(\mathcal{O}(\mathcal{S})) = 1$$

  - A condition on the Euler number c.f. that of the base:

$$\chi(\mathcal{S}) \geq \chi(\mathcal{B})$$

  - A condition following from birationality to the base (see Morrison, Park, JHEP 1210 (2012) 128):

$$\mathcal{S} \cdot \mathcal{S} \cdot D_\alpha = -c_1(\mathcal{B}) \cdot S \cdot D_\alpha$$

- Koszul derivation of second condition as example:

$$0 \to \mathcal{O}(-\mathcal{S}) \to \mathcal{O} \to \mathcal{O}|_{\mathcal{S}} \to 0$$

| | | | |
|---|---|---|---|
| $h^0$ | ? | $1$ | $1$ |
| $h^1$ | ? | $0$ | $0$ |
| $h^2$ | ? | $0$ | $0$ |
| $h^3$ | ? | $1$ | $0$ |

$$\Rightarrow h^3(\mathcal{X}, \mathcal{O}(-S)) = 1$$

$$\Rightarrow h^0(\mathcal{X}, \mathcal{O}(S)) = 1$$

- For $\left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 \\ \mathbb{P}^2 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^2 & 1 & 0 & 1 & 1 \end{array}\right]$ , for example we find the following section: $\mathcal{O}(\mathcal{S}) = \mathcal{O}(-1,1,0,1)$
- Build the explicit description of the section (remember $h^0(\mathcal{O}(\mathcal{S})) = 1$ ) in the same way we built gCICYs.
- Now we have an explicit section we can put the fibration in Weierstrass form using the Deligne procedure.
    - (see Ovrut, Pantev and Park, JHEP 0005 (2000) 045)

- Idea:

$$z \in H^0(\mathcal{X}, \mathcal{S}) \qquad h^0(\mathcal{X}, \mathcal{S}) = 1$$

$$x \in H^0(\mathcal{X}, \mathcal{S}^2 \otimes K_{\mathcal{B}}^{-2})$$

$$h^0(\mathcal{X}, \mathcal{S}^2 \otimes K_{\mathcal{B}}^{-2}) = 29$$

$$y \in H^0(\mathcal{X}, \mathcal{S}^3 \otimes K_{\mathcal{B}}^{-3})$$

$$h^0(\mathcal{X}, \mathcal{S}^3 \otimes K_{\mathcal{B}}^{-3}) = 66$$

- Then get (Weierstrass) relation between them in:

$$W \in H^0(\mathcal{X}, \mathcal{S}^6 \otimes K_{\mathcal{B}}^{-6})$$

# What do the theories look like:

- M-Theory:
  - 3 Vector multiplets
  - 48 Hyper multiplets
- F-theory 1:
  - $SU(2) \times U(1)$ gauge group
  - 0 Tensor multiplets
  - 4 Vector multiplets
  - 277 Hyper multiplets (48 Neutral)
- F-theory 2:
  - $U(1)$ gauge group
  - 1 Tensor multiplet
  - 1 Vector multiplet
  - 245 Hyper multiplets (48 Neutral)

- As a slightly more non-trivial example, consider the following configuration matrix:

$$X_3^{\mathbb{E}_1} = \left[\begin{array}{c|c:cccccccc} \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right] .$$

- This admits nine obvious genus one fibrations…

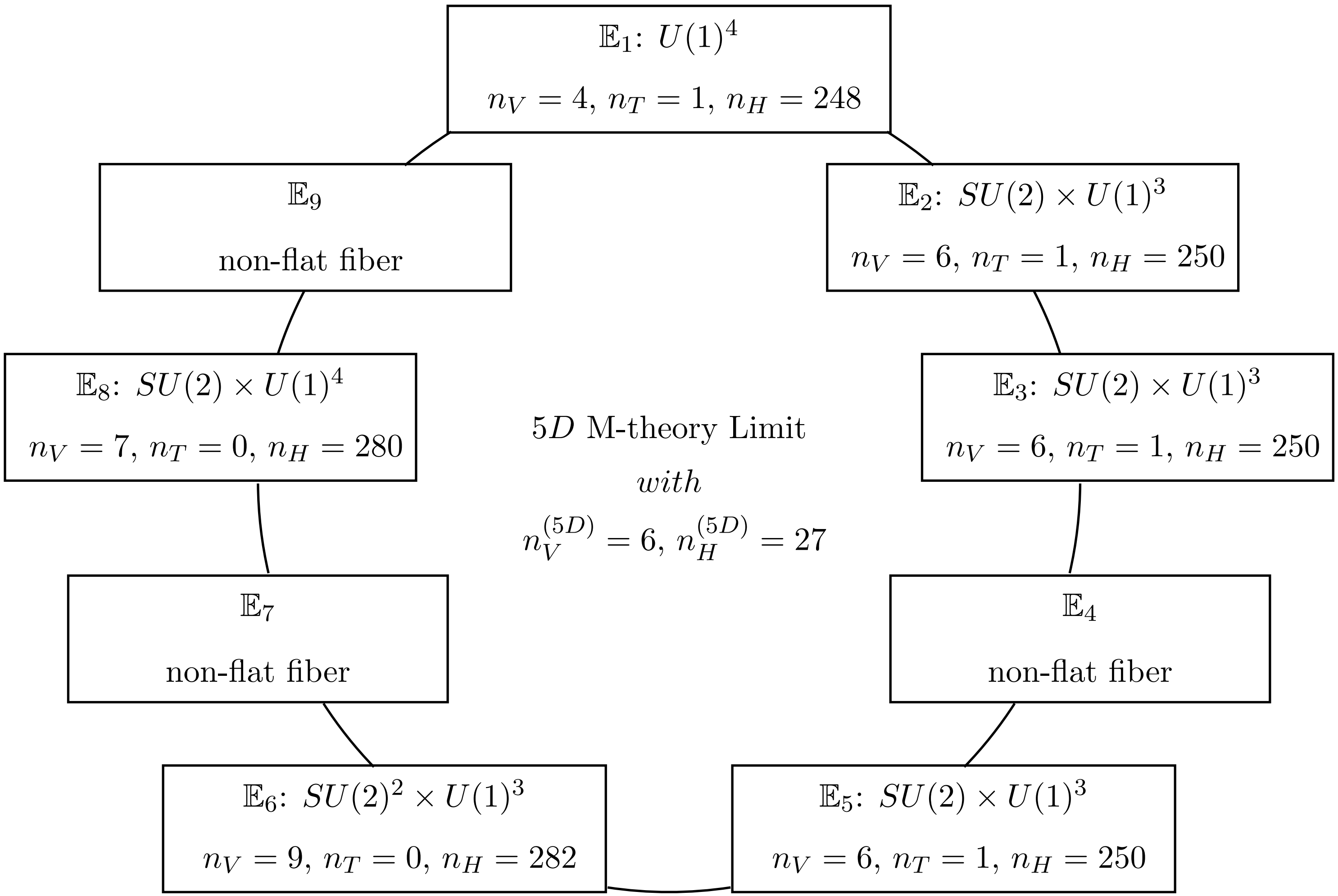



**Figure 7:** *F-theory models in 6D with the same 5D M-theory limit where* $n_V^{(5D)} = 6$ *and* $n_H^{(5D)} = 27$.