Bonus Material:
Chow Groups, Fluxes, Matter

- arXiv:1402.5144 with **M. Bies, C. Mayrhofer, C. Pehle**
- arXiv:1706.04616 **M. Bies, C. Mayrhofer**

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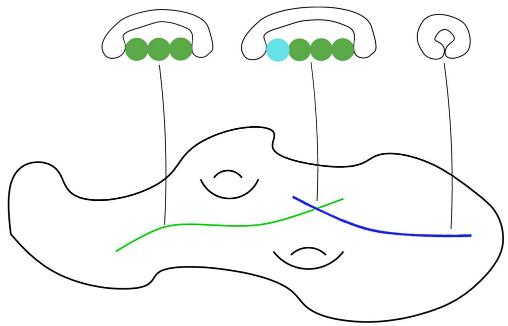
Defining the gauge background

Massless matter $\leftrightarrow$ wrapped M2

M2-branes couple to M-theory
3-form gauge potential C_3:



- $C_3 \simeq C_3 + d\Lambda_2$ is higher form gauge potential
- $G_4 = dC_3$ field strength $\longrightarrow$ flux $[G_4] \in H^{2,2}(\hat{Y}_4)$

Chiral index:

$$\nu_+(\mathbf{R}) - \nu_-(\mathbf{R}) = \int_{S^a(\mathbf{R})} [G_4] \qquad\qquad S^a(\mathbf{R}) \text{ : matter surface}$$

[Donagi,Wijnholt'09]; [Braun,Collinucci,Valandro] [Marsano,Schäfer-Nameki]

[Krause,Mayrhofer,TW], [Grimm,Hayashi]'11; [Collinucci, Savelli '10/12], ...

- What specifies the C_3 'gauge data' beyond the field strength/flux?
- What counts the actual number of $\mathcal{N} = 1$ chiral multiplets?

Gauge background in F/M-theory

Wanted: refinement of gauge data beyond 4-form field strength

1) Theoretical Description

$$0 \longrightarrow \underbrace{J^2(\hat{Y}_4)}_{\oint C_3 \text{ 'Wilson lines'}} \longrightarrow \underbrace{H^4_D(\hat{Y}_4, \mathbb{Z}(2))}_{\textbf{Deligne cohomology}} \xrightarrow[\textbf{onto}]{\hat{c}_2} \underbrace{H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)}_{\text{field strength } G_4} \longrightarrow 0$$

- Deligne cohomology group $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ $\longleftrightarrow$ equ. cl. of gauge background
- $H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)$ $\longleftrightarrow$ field strength G_4
- Intermediate Jacobian $\longleftrightarrow$ Wilson lines $\oint C_3$

$$\mathcal{J}^2(\hat{Y}_4) \simeq H^3(\hat{Y}_4, \mathbb{C})/(H^{3,0}(\hat{Y}_4) + H^{2,1}(\hat{Y}_4) + H^3(\hat{Y}_4, \mathbb{Z}))$$

$H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ in F/M-theory context: [Donagi,Curio'98] [Donagi,Wijnholt'12/13] [Anderson,Heckman,Katz'13]

Gauge background in F/M-theory

2) Practical Parametrisation on smooth $\hat{Y}_4$

Analogy: Line bundle

$$0 \to \underbrace{\mathcal{J}^1(X)}_{\text{'Wilson lines'} \oint A} \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} \underbrace{H^{1,1}_{\mathbb{Z}}(X)}_{\text{'field strength'} F} \longrightarrow 0$$

line bundle group Pic(X) $\xleftrightarrow[1:1]{L=\mathcal{O}(-D)}$ Group of divisor classes

Wanted:

Deligne $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ $\xleftrightarrow[???]{}$???

Gauge background in F/M-theory

2) Practical Parametrisation on smooth $\hat{Y}_4$

line bundle group Pic(X) $\xleftrightarrow[1:1]{L=\mathcal{O}(-D)}$ Group of divisor classes

Wanted:

Deligne $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ $\xleftrightarrow[???]{}$???

Analogous geometric concept: [Chow 1956]

Group of algebraic $2_{\mathbb{C}}$-cycles modulo rational equivalence $=$ CH$_2(X)$

- Rational equivalence: $C_1 \cong C_2 \in Z_p(X)$ if $C_1 - C_2$ is zero/pole of a meromorphic function defined on an $(p+1)$-dimensional irreducible subvariety of X
- Chow group $\mathrm{CH}^k(X)$ $=$ group of rational equivalence classes of codim. k-cycles
- Special case: $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$ (Recall: assuming smoothness!)

Gauge background in F/M-theory

Idea: [Bies,Mayrhofer,Pehle,TW'14]

Concrete representation of gauge data by specification of element

$$A \in \mathrm{CH}^2(\hat{Y}_4) \text{ with } [A] = G_4 \in H^{2,2}(\hat{Y}_4)$$

via 'refined cycle map' $\hat{\gamma}_2 : \mathrm{CH}^2(\hat{Y}_4) \to H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ e.g. [Esnault,Viehweg'88]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}^2_{\text{hom}}(\hat{Y}_4) & \longrightarrow & \overbrace{\mathbf{CH^2(\hat{Y}_4)}}^{\textbf{geometry}} & \xrightarrow{\gamma_2} & H^{2,2}_{alg}(\hat{Y}_4) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle \hat{\gamma}_2} & & \downarrow & & \\
0 & \longrightarrow & \underbrace{J^2(\hat{Y}_4)}_{\oint C_3 \text{ 'Wilson lines'}} & \longrightarrow & \underbrace{\mathbf{H^4_D(\hat{Y}_4, \mathbb{Z}(2))}}_{\textbf{full gauge data}} & \xrightarrow{\hat{c}_2} & \underbrace{H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)}_{\text{field strength } G_4} & \longrightarrow & 0
\end{array}$$

Properties of $\hat{\gamma}_2$ of importance for us:

1. Changing cycles up to rational equiv. does not change gauge data!
2. Combining with $\hat{c}_2$ gives flux: $G_4(A) = \hat{c}_2 \circ \hat{\gamma}_2(A) \in H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)$

Gauge background in F/M-theory

$$
\begin{array}{ccccccccc}
 & & & & \overbrace{\mathrm{CH}^{\mathbf{2}}(\hat{\mathbf{Y}}_{\mathbf{4}})}^{\textbf{geometry}} & & & & \\
0 & \longrightarrow & \mathrm{CH}^2_{\mathsf{hom}}(\hat{Y}_4) & \longrightarrow & \mathrm{CH}^{\mathbf{2}}(\hat{\mathbf{Y}}_{\mathbf{4}}) & \xrightarrow{\gamma_2} & H^{2,2}_{alg}(\hat{Y}_4) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle \hat{\gamma}_{\mathbf{2}}} & & \downarrow & & \\
0 & \longrightarrow & \underbrace{J^2(\hat{Y}_4)}_{\oint C_3\ '\text{Wilson lines}'} & \longrightarrow & \underbrace{\mathbf{H}^{\mathbf{4}}_{\mathbf{D}}(\hat{\mathbf{Y}}_{\mathbf{4}}, \mathbb{Z}(\mathbf{2}))}_{\textbf{full gauge data}} & \xrightarrow{\hat{c}_2} & \underbrace{H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)}_{\text{field strength } G_4} & \longrightarrow & 0
\end{array}
$$

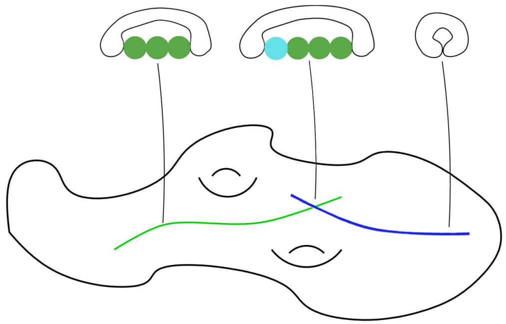
Further properties:

- $\hat{\gamma}_2$ is surjective (over $\mathbb{Q}$) iff the Hodge conjecture holds
- $\hat{\gamma}_2$ is in general not injective, i.e. different elements in $\mathrm{CH}^2(\hat{Y}_4)$ may give same gauge data.

A cohomology formula

Aim: Extract bundle data relevant for matter representation R

- Fix element $A \in \mathrm{CH}^2(\hat{Y}_4)$ with $[A] = G_4 \in H^{2,2}(\hat{Y}_4)$
- Matter surface $\pi_{\mathbf{R}} : S^a(\mathbf{R}) \to C_{\mathbf{R}}$
- $S^a(\mathbf{R}) \cap A \in \mathrm{CH}^2(S^a(\mathbf{R}))$
  $=$ Chow class of points on $S^a(\mathbf{R})$



- Projection to base B_3 gives points on matter curve C_R:

$$A|_{\mathbf{R}} := \pi_{\mathbf{R}*}(S^a(\mathbf{R}) \cap A) \in \mathrm{CH}^1(C_{\mathbf{R}}) \cong \mathrm{Pic}(C_R)$$

- Pick a representative $A|_{\mathbf{R}} \in Z_0(C_{\mathbf{R}})$ and $L = \mathcal{O}_{C_{\mathbf{R}}}(A|_{\mathbf{R}})$
- Massless $\mathcal{N} = 1$ chiral multiplets counted by [Bies,Mayrhofer,(Pehle),TW'14 (17)]

$$H^i(C_R, L \otimes \sqrt{K_{C_{\mathbf{R}}}}), \qquad i = 0, 1$$

✓ chiral index $\chi(\mathbf{R}) = \deg(L) = [S^a(\mathbf{R})] \cdot [A] \equiv \int_{S^a(\mathbf{R})} G_4$

A cohomology formula

Justification:

- Zero modes in $\beta^a(\mathbf{R})$ come from quantization of the moduli space of M2 wrapped on fibre of $S^a(\mathbf{R})$ [Witten'97]
- Gauge background on $C_{\mathbf{R}}$ from integrating C_3 over fibre is mathematically given precisely by operation of intersection and projection

$$A|_{\mathbf{R}} := \pi_{\mathbf{R}*}(S^a(\mathbf{R}) \cap A)$$

- All operations are compatible with rational equivalence and hence, by the refined cycle map, with the definition of the gauge background.

Open questions

Generalization to singular Weierstrass model?

- Crepant resolution may not exist ($\mathbb{Q}$-factorial terminal singularities!)
- Resolution incompatible with non-abelian gauge data
  $\rightarrow$ require generalisation to non-abelian backgrounds/
  T-branes in global setups

Alternative approaches: [Collinucci,Savelli'14], [Anderson,Heckman,Katz], . . .