Zoology of complex threefold bases in F-theory

1510.04978 & 1710.11235 with W. Taylor

Yi-Nan Wang

CTP, MIT

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The setup and goal

• Classify and characterize topologically distinct smooth projective base threefolds B such that there exists an elliptic Calabi-Yau fourfold X over B.

• We assume that the elliptic fibration has a global holomorphic section, so that we write X in the Weierstrass form

$$y^2 = x^3 + fx + g, \tag{1}$$

f and g are holomorphic sections of line bundles $-4K_B$ and $-6K_B$.

• Usually the Weierstrass form is singular, e. g.

$$y^2 = x^3 + u^2x + u^2, \tag{2}$$

so we need to perform a crepant resolution $\hat{X} \to X$.
After the resolution, $\hat{X}$ may still have terminal singularities in cod-3 or cod-4.

The setup and goal

- The cases where B does not satisfy these conditions:

(1) The anticanonical line bundle $-K_B$ is not effective, hence $f = g = 0$.

(2) For any elliptic fibration over B, f and g vanishes to order (4,6) or higher on a cod-1 locus, e. g.

$$y^2 = x^3 + u^4 x + u^6 \tag{3}$$

- For the valid bases, we classify them into "resolvable base" and "good base". (Taylor, Wang 17')

(1) Resolvable base: there exists a curve $C \in B$, such that for any elliptic fibration over B, f and g vanish to order (4,6) or higher on C. But one can perform a series of blow ups and transform B into a good base.

(2) Good base: there does not exist such a (4,6) curve.

Two types of bases

- Example of a generic elliptic fibration over a resolvable base B:

$$y^2 = x^3 + (u^4 + v^4)x + (u^6 + v^6) \tag{4}$$

f and g vanish to order $(4, 6)$ on $u = v = 0$.

The singularity at $x = y = u = v = 0$ can be resolved by a weighted blow up on the ambient space:

$$y \to y\xi^3 \ , \ x \to x\xi^2 \ , \ u \to u\xi \ , \ v \to v\xi. \tag{5}$$

This resolution is actually equivalent to blowing up the base curve $u = v = 0$ on B.

If this resolution is not performed, sometimes there will be a non-flat fiber in $\hat{X}$ (tensionless string in F-theory effective theory).

- Good base: no such blow up of base is required.

Characterization of good base

For a good base B without (4,6)-curve, we can characterize it by looking at the generic elliptic CY4 X over B.

• Generic: f and g are generic holomorphic sections of $-4K_B$ and $-6K_B$, such that the discriminant $\Delta = 4f^3 + 27g^2$ vanishes to lowest order on each cod-1 locus on B.

• Equivalently, the gauge group in F-theory description is minimal: non-Higgsable gauge groups; the number of complex structure moduli is maximal.

• Examples: $B = \mathbb{P}^3$ or Fano/weak Fano threefolds$\rightarrow$ no non-Higgsable gauge groups.

• Generalized Hirzebruch threefolds $\tilde{\mathbb{F}}_{18}$: $\mathbb{P}^1$ bundle over $\mathbb{P}^2$ with twist $-18H$. (f, g, Δ) vanishes to order $(4, 5, 10) \rightarrow E_8$.

$$h^{1,1}(X) = h^{1,1}(B) + \mathrm{rk}(G) + 1. \quad (6)$$

General classification of bases

- General classification of base B that supports an elliptic CY:

- For elliptic CY3, $\dim(B) = 2$, B is either rational surface or Enriques surface (Grassi 91')

- For elliptic CY4, $\dim(B) = 3$, no known mathematical result.

Nonetheless, we can try to explore a set of threefold bases that are easy to study: rational threefolds that can be birationally transformed to $\mathbb{P}^3$ via a series of blow up/down.

Every threefold in this sequence of blow up/down is smooth (different from the usual Mori theory).

Toric bases

The easiest subset of bases we can study is the toric bases.

A toric threefold B is described by a fan Σ which is a collection of 3D, 2D, 1D simplicial cones in the lattice $N = \mathbb{Z}^3$.

1D rays$\leftrightarrow$ toric divisors, 2D cones$\leftrightarrow$toric curves, 3D cones$\leftrightarrow$toric points.

(1) Effective cone and Mori cone are generated by the set of 1D rays and 2D cones respectively

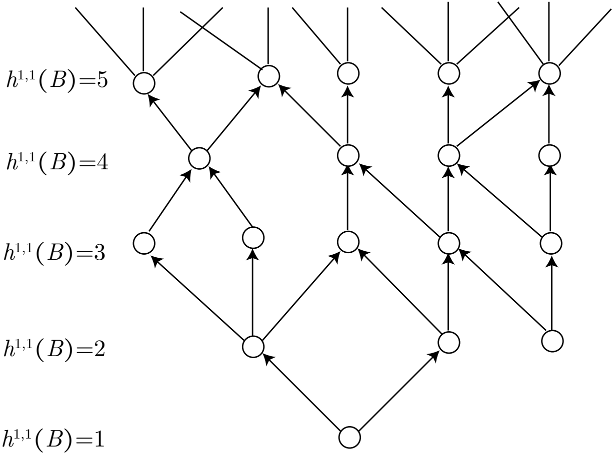
(2) The generators of effective line bundles can be easily read out by looking at the points in the dual lattice M of N

(3) Blow up/down to another toric threefold is simply described by adding/deleting 1D ray and changing the cones

A directed graph with layer structure

Nodes: smooth toric threefold bases

Links: blow up/down



Approach 1

Random walk on the toric threefold landscape (Taylor, Wang 15')

- Start from $\mathbb{P}^3$, do a random sequence of 100,000 blow up/downs.
- Never pass through bases with (4,6) curves (excluding E_8 gauge group).
- In total 100 runs.

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SU(2)	SU(3)	G_2	SO(7)
13.6	2.0	9.7	4×10^{-6}
SO(8)	F_4	E_6	E_7
1.0	2.8	0.3	0.2

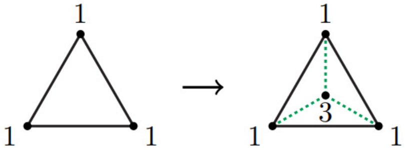
Average number of non-Higgsable gauge group on a base.

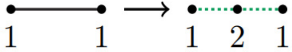
- 76% of bases have $SU(3) \times SU(2)$ non-Higgsable cluster.
- Total number $\sim 10^{48}$. $\max(h^{1,1}(B)) \sim 120$.

Approach 2

Combinatorially generate toric threefold bases by blowing up Fano bases (Halverson, Long, Sung 17')

- Put additional "height constraint" during the blow up process: $h \leq 6$.





- Blow ups of points before blow ups of curves.
- Generally allow (4,6) curves.
- Rigorously proved that $N \geq \frac{4}{3} \times 2.96 \times 10^{755}$ bases.

New One-way Monte Carlo approach

(Taylor, Wang 17')

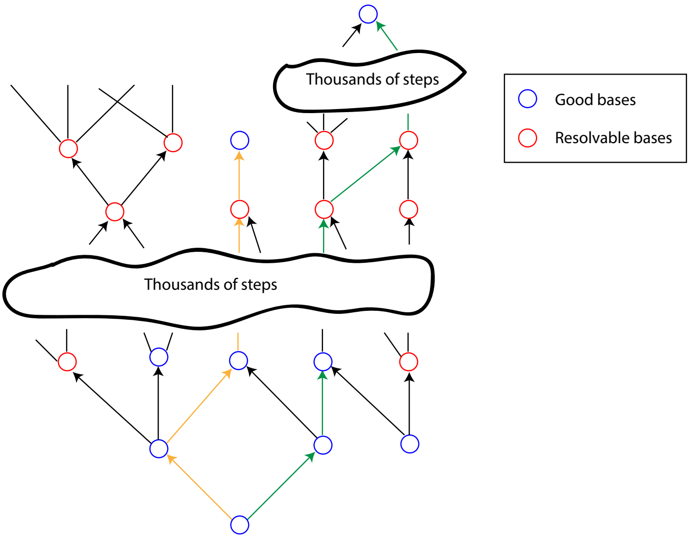
• We want to include all the resolvable bases in our directed graph and draw all the edges between them. We also want to generate some good bases in this process.

• In this approach, we cannot perform a random walk, because the good bases are extremely rare among resolvable bases.

• We do a random sequence of blow ups starting from a single base, e.g. $\mathbb{P}^3$, until we hit the end point where any blow up will lead to cod-1 (4,6). At each step, the possibility of choosing each outgoing path is equal.

• According to the definition, the end point is always good. But most of the bases between $h^{1,1}(B) \sim 10$ and the end point are only resolvable.

New One-way Monte Carlo approach



Results

In total, we generated 2,000 random blow up sequences starting from $\mathbb{P}^3$.

• The end points are concentrated at certain layers. For example, 15% percent of branches end on layer 2249 and 15% percent of branches end on layer 2303. But there's nothing between them.

• End points are highly non-random.

• The gauge groups on end point bases are $SU(2)^a \times G_2^b \times F_4^c \times E_8^d \times H$, where

$$a \cong \left[\frac{h^{1,1}(B)+1}{6}\right],\ b \cong \left[\frac{h^{1,1}(B)+1}{9}\right],\ c \cong \left[\frac{h^{1,1}(B)+1}{24}\right],\ d \cong \left[\frac{h^{1,1}(B)}{68}\right] \quad (7)$$

H is some other gauge group that rarely appears. For example, if the end point is on layer 2999, then $H =$SU(3).

• For different end point bases on the same layer, they have same non-Higgsable gauge groups but the their adjacency are different.

Results

- After computing $h^{1,1}(X)$, $h^{3,1}(X)$ of generic elliptic CY4 X over the end point bases B, we found that they resemble the mirror of simple elliptic CY4s over simple bases.

(1) For the bases with $h^{1,1}(B) = 2303$, $h^{1,1}(X) = 3878$, $h^{3,1}(X) = 2$: mirror of generic elliptic CY4 over $\mathbb{P}^3$.

(2) For the bases with $h^{1,1}(B) = 2591$, $h^{1,1}(X) = 4358$, $h^{3,1}(X) = 3$: mirror of generic elliptic CY4 over generalized Hirzebruch threefold $\tilde{\mathbb{F}}_3$.

- Maybe easy to compute the Gukov-Vafa-Witten potential on these geometry because they have simple mirror CY4s.

We also tried other starting points such as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and generalized Hirzebruch threefold $\tilde{\mathbb{F}}_2$, and it seems that we get the same classes of end point bases.

- The total number of resolvable bases $\gtrsim 10^{3046}$, the total number of good bases $\gtrsim 10^{253}$.

Structure of roots

• To really complete the story, we need to make a list of starting points (roots in the directed graph).

• To get a feeling of the abundance of roots, we try to randomly blow down an end point base (reverse the orientation of edge). Can we get $\mathbb{P}^3$?

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• To really complete the story, we need to make a list of starting points (roots in the directed graph).

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• It turns out that we will be stuck at some "exotic starting point" base with 50-100 toric rays which cannot be further blown down to get another smooth toric base.

• Consistent with Mori theory.

• Estimate the number of these exotic starting points? Allowing singular bases?

Non-toric threefolds

(1) Effective cone and Mori cone are not easy to compute.

(2) The generators of effective line bundles are even harder to write down (e.g. del Pezzo surface).

(3) The number of possible blow ups on a threefold base is not clear.

• A problem: given a smooth threefold B, write down all the distinct classes of irreducible curves on it (write down all the components of the Hilbert scheme of curves that correspond to irreducible curves), which we can blow them up and get non-isomorphic threefold bases.

• The problem is unsolved even for $B = \mathbb{P}^3$. For example, there are two types of degree 3 curves:

(1) $3H \bigcap H$: plane cubic curve with genus 1.

(2) twisted cubic curve (cubic rational curve) with genus 0.

Derive non-Higgsable gauge group

• One step towards understanding non-toric threefold is studying the non-Higgsable gauge groups.

• In 6D F-theory, $\dim(B) = 2$, the gauge groups can be easily read out with the intersection numbers of curves on B.

• e.g. an isolated (-3/-4/-5/-6/-8/-12) curve will give SU(3), SO(8), F_4, E_6, E_7, E_8 gauge groups.

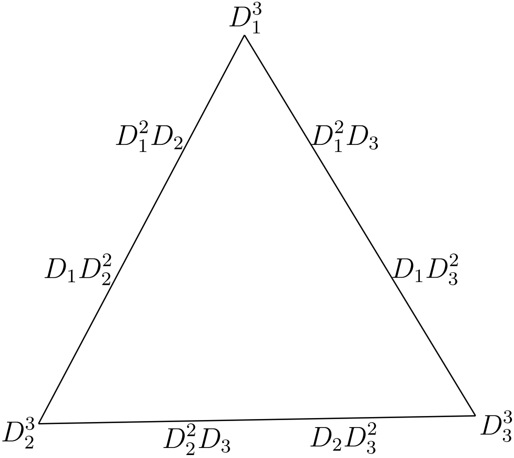
• How to read out the non-Higgsable gauge groups in 4D F-theory using the local geometric data on the base B?

• Formula using canonical class, normal bundle and intersection relations of divisors (Morrison Taylor 14'). Hard to actually apply.

• No formula with triple intersection numbers as input; no classification of 4D NHC.

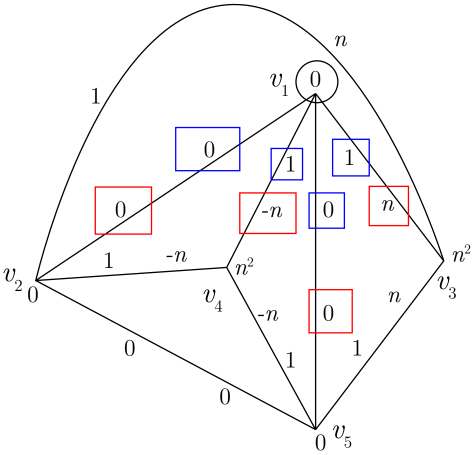
Derive non-Higgsable gauge group

- We introduce a diagram to present the triple intersection numbers on a smooth toric threefold.



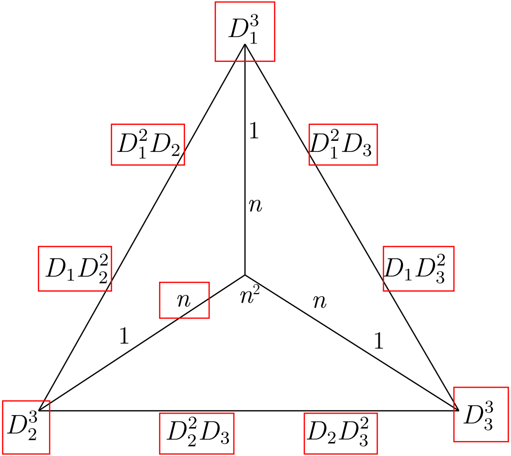
Derive non-Higgsable gauge group

Generalized Hirzebruch threefold $\tilde{\mathbb{F}}_n$.



Derive non-Higgsable gauge group

- Trying to derive the non-Higgsable gauge group on a $\mathbb{P}^2$ divisor using the following numbers:



Derive non-Higgsable gauge group

- Using supervised machine learning.

Input: the set of triple intersection numbers near a $\mathbb{P}^2$ divisor D.

Output: the non-Higgsable gauge group on D.

- Generate a decision tree to predict the gauge group, easy to formulate analytic conjectures.