Invertible braided module categories and graded braided extensions of fusion categories

Dmitri Nikshych (joint work with Alexei Davydov)

University of New Hampshire

October 15, 2018

Outline

Outline

1. Graded extensions of fusion categories
2. Braided module categories over braided fusion categories
3. Braided extensions

We work over an algebraically closed field k.

We work over an algebraically closed field k. Let G be a finite group.

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

Classification of extensions (Etingof-N-Ostrik) via higher cat groups

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

Classification of extensions (Etingof-N-Ostrik) via higher cat groups

- { Groupoid of G-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of monoidal 2-functors $\boxed{G \to \mathrm{BrPic}(\mathcal{B})}$} (invertible $\mathcal{B}$-bimodule categories).

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

Classification of extensions (Etingof-N-Ostrik) via higher cat groups

- { Groupoid of G-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of monoidal 2-functors $\boxed{G \to \mathrm{BrPic}(\mathcal{B})}$} (invertible $\mathcal{B}$-bimodule categories).
- Equivalently: monoidal functors with a vanishing obstruction in $H^4(G, k^\times)$ (parameterization by a torsor over $H^3(G, k^\times)$).

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

Classification of extensions (Etingof-N-Ostrik) via higher cat groups

- { Groupoid of G-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of monoidal 2-functors $\boxed{G \to \mathrm{BrPic}(\mathcal{B})}$} (invertible $\mathcal{B}$-bimodule categories).
- Equivalently: monoidal functors with a vanishing obstruction in $H^4(G, k^\times)$ (parameterization by a torsor over $H^3(G, k^\times)$).
- Equivalently: homotopy classes of maps $BG \to B\mathrm{BrPic}(\mathcal{B})$.

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

Classification of extensions (Etingof-N-Ostrik) via higher cat groups

- { Groupoid of G-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of monoidal 2-functors $\boxed{G \to \mathrm{BrPic}(\mathcal{B})}$} (invertible $\mathcal{B}$-bimodule categories).
- Equivalently: monoidal functors with a vanishing obstruction in $H^4(G, k^\times)$ (parameterization by a torsor over $H^3(G, k^\times)$).
- Equivalently: homotopy classes of maps $BG \to B\mathrm{BrPic}(\mathcal{B})$.
- For braided $\mathcal{B}$ { G-crossed graded extensions of $\mathcal{B}$ } $\simeq$ { monoidal 2-functors $G \to \mathrm{Pic}(\mathcal{B})$} (invertible $\mathcal{B}$-module categories).

We work over an algebraically closed field k. Let G be a finite group.

A *G-graded* fusion category is $\mathcal{C} = \oplus_{x \in G} \mathcal{C}_x$ with $\otimes : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$. If $\mathcal{C}_1 = \mathcal{B}$ we say that $\mathcal{C}$ is a *G-extension* of $\mathcal{B}$.

Classification of extensions (Etingof-N-Ostrik) via higher cat groups

- { Groupoid of G-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of monoidal 2-functors $\boxed{G \to \mathrm{BrPic}(\mathcal{B})}$} (invertible $\mathcal{B}$-bimodule categories).
- Equivalently: monoidal functors with a vanishing obstruction in $H^4(G, k^\times)$ (parameterization by a torsor over $H^3(G, k^\times)$).
- Equivalently: homotopy classes of maps $BG \to B\mathrm{BrPic}(\mathcal{B})$.
- For braided $\mathcal{B}$ { G-crossed graded extensions of $\mathcal{B}$ } $\simeq$ { monoidal 2-functors $G \to \mathrm{Pic}(\mathcal{B})$} (invertible $\mathcal{B}$-module categories).

$\mathrm{BrPic}(\mathcal{B})$ is a *2-categorical group*. It determines the homotopy class of a topological space (a *3-type*) with $\pi_1 = \mathrm{BrPic}(\mathcal{B})$, $\pi_2 = \mathrm{Inv}(\mathcal{Z}(\mathcal{B}))$, and $\pi_3 = k^\times$.

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

$\{$ Groupoid of braided A-extensions of $\mathcal{B}$ $\} \simeq \{$ groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}\}$.

Here $\mathbf{Pic_{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

$\{$ Groupoid of braided A-extensions of $\mathcal{B}$ $\} \simeq \{$ groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}$ $\}$.

Here $\mathbf{Pic_{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Plan of the talk

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

{ Groupoid of braided A-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}$ }.

Here $\mathbf{Pic_{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Plan of the talk

- Describe the structures involved

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

$\{$ Groupoid of braided A-extensions of $\mathcal{B}$ $\} \simeq \{$ groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic}_{\mathbf{br}}(\mathcal{B})}$ $\}$.

Here $\mathbf{Pic}_{\mathbf{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Plan of the talk

- Describe the structures involved
- Prove the classification result

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

$\{$ Groupoid of braided A-extensions of $\mathcal{B}$ $\} \simeq \{$ groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}$ $\}$.

Here $\mathbf{Pic_{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Plan of the talk

- Describe the structures involved
- Prove the classification result
- Explain relevant higher categories and functors algebraically (using the language of obstructions)

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

$\{$ Groupoid of braided A-extensions of $\mathcal{B}$ $\} \simeq \{$ groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}\}$.

Here $\mathbf{Pic_{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Plan of the talk

- Describe the structures involved
- Prove the classification result
- Explain relevant higher categories and functors algebraically (using the language of obstructions)
- Compute braided Picard groups in interesting cases

Problem

Let A be a finite abelian group. Let $\mathcal{B}$ be a braided fusion category. Classify braided A-extensions of $\mathcal{B}$.

Solution

{ Groupoid of braided A-extensions of $\mathcal{B}$ } $\simeq$ { groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}$ }.

Here $\mathbf{Pic_{br}}(\mathcal{B})$ consists of invertible *braided* $\mathcal{B}$-bimodule categories), we call it the *braided* Picard group of $\mathcal{B}$.

Plan of the talk

- Describe the structures involved
- Prove the classification result
- Explain relevant higher categories and functors algebraically (using the language of obstructions)
- Compute braided Picard groups in interesting cases

Explanation of terms

Explanation of terms

Let $\mathcal{M}$ be a right $\mathcal{B}$-module category and $\mathcal{N}$ be a left $\mathcal{B}$-module category.

Explanation of terms

Let $\mathcal{M}$ be a right $\mathcal{B}$-module category and $\mathcal{N}$ be a left $\mathcal{B}$-module category.

The $\mathcal{B}$-module tensor product $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$

consists of pairs $(V \in \mathcal{M} \boxtimes \mathcal{N},\, \gamma = \{\gamma_X\})$, where the middle balancing

$$\gamma_X : V \otimes (X \boxtimes 1) \to (1 \boxtimes X) \otimes V, \qquad X \in \mathcal{B}$$

is associative.

Explanation of terms

Let $\mathcal{M}$ be a right $\mathcal{B}$-module category and $\mathcal{N}$ be a left $\mathcal{B}$-module category.

The $\mathcal{B}$-module tensor product $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$

consists of pairs $(V \in \mathcal{M} \boxtimes \mathcal{N},\, \gamma = \{\gamma_X\})$, where the middle balancing

$$\gamma_X : V \otimes (X \boxtimes 1) \to (1 \boxtimes X) \otimes V, \qquad X \in \mathcal{B}$$

is associative. This is similar to tensor product of modules over a ring.

Explanation of terms

Let $\mathcal{M}$ be a right $\mathcal{B}$-module category and $\mathcal{N}$ be a left $\mathcal{B}$-module category.

The $\mathcal{B}$-module tensor product $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$

consists of pairs $(V \in \mathcal{M} \boxtimes \mathcal{N},\, \gamma = \{\gamma_X\})$, where the middle balancing

$$\gamma_X : V \otimes (X \boxtimes 1) \to (1 \boxtimes X) \otimes V, \qquad X \in \mathcal{B}$$

is associative. This is similar to tensor product of modules over a ring. If $\mathcal{M}$, $\mathcal{N}$ are bimodule categories then so is $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$.

$\mathcal{B}$-Bimod is a *monoidal 2-category* via $\boxtimes_{\mathcal{B}}$

Objects = $\mathcal{B}$-bimodule categories, 1-cells = $\mathcal{B}$-bimodule functors, 2-cells = $\mathcal{B}$-bimodule natural transformations.

Explanation of terms

Let $\mathcal{M}$ be a right $\mathcal{B}$-module category and $\mathcal{N}$ be a left $\mathcal{B}$-module category.

The $\mathcal{B}$-module tensor product $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$

consists of pairs $(V \in \mathcal{M} \boxtimes \mathcal{N},\, \gamma = \{\gamma_X\})$, where the middle balancing

$$\gamma_X : V \otimes (X \boxtimes 1) \to (1 \boxtimes X) \otimes V, \qquad X \in \mathcal{B}$$

is associative. This is similar to tensor product of modules over a ring. If $\mathcal{M}$, $\mathcal{N}$ are bimodule categories then so is $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$.

$\mathcal{B}$-**Bimod** is a *monoidal 2-category* via $\boxtimes_{\mathcal{B}}$

Objects $=$ $\mathcal{B}$-bimodule categories, 1-cells $=$ $\mathcal{B}$-bimodule functors, 2-cells $=$ $\mathcal{B}$-bimodule natural transformations.

We will suppress the assocativity 2-cells for $\boxtimes_{\mathcal{B}}$.

Explanation of terms

Let $\mathcal{M}$ be a right $\mathcal{B}$-module category and $\mathcal{N}$ be a left $\mathcal{B}$-module category.

The $\mathcal{B}$-module tensor product $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$

consists of pairs $(V \in \mathcal{M} \boxtimes \mathcal{N},\, \gamma = \{\gamma_X\})$, where the middle balancing

$$\gamma_X : V \otimes (X \boxtimes 1) \to (1 \boxtimes X) \otimes V, \qquad X \in \mathcal{B}$$

is associative. This is similar to tensor product of modules over a ring. If $\mathcal{M}$, $\mathcal{N}$ are bimodule categories then so is $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$.

$\mathcal{B}$-**Bimod** is a *monoidal 2-category* via $\boxtimes_{\mathcal{B}}$

Objects = $\mathcal{B}$-bimodule categories, 1-cells = $\mathcal{B}$-bimodule functors, 2-cells = $\mathcal{B}$-bimodule natural transformations.

We will suppress the assocativity 2-cells for $\boxtimes_{\mathcal{B}}$.

The Brauer-Picard categorical 2-group BrPic($\mathcal{B}$) is the "pointed part" of $\mathcal{B}$-**Bimod**

Objects are invertible w.r.t $\boxtimes_{\mathcal{B}}$, all cells are isomorphisms.

Outline

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_{\pm}^{\mathcal{M}} : \mathcal{B} \to \text{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi):

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_{\pm}^{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_{\pm}^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_\pm^{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_\pm^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)
- Since $\mathcal{M}$ is a $\mathcal{B} - \mathrm{End}_{\mathcal{B}}(\mathcal{M})$-bimodule, one can turn $\mathcal{M}$ into a $\mathcal{B}$-bimodule category in 2 different ways: $\mathcal{M}_\pm$ (using $\alpha_\pm^{\mathcal{M}}$).

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_{\pm}^{\mathcal{M}} : \mathcal{B} \to \text{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_{\pm}^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)
- Since $\mathcal{M}$ is a $\mathcal{B} - \text{End}_{\mathcal{B}}(\mathcal{M})$-bimodule, one can turn $\mathcal{M}$ into a $\mathcal{B}$-bimodule category in 2 different ways: $\mathcal{M}_{\pm}$ (using $\alpha_{\pm}^{\mathcal{M}}$).
- Two monoidal 2-categories: $\mathcal{B}$-**Mod**$_{\pm}$ with products $\mathcal{M}_{\pm} \boxtimes_{\mathcal{B}} \mathcal{N}$.

Relation between $\pm$ products:

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_{\pm}^{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_{\pm}^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)
- Since $\mathcal{M}$ is a $\mathcal{B} - \mathrm{End}_{\mathcal{B}}(\mathcal{M})$-bimodule, one can turn $\mathcal{M}$ into a $\mathcal{B}$-bimodule category in 2 different ways: $\mathcal{M}_{\pm}$ (using $\alpha_{\pm}^{\mathcal{M}}$).
- Two monoidal 2-categories: $\mathcal{B}$-**Mod**$_{\pm}$ with products $\mathcal{M}_{\pm} \boxtimes_{\mathcal{B}} \mathcal{N}$.

Relation between $\pm$ products:

There is natural equivalence $\mathcal{N}_{-} \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}_{+} \boxtimes_{\mathcal{B}} \mathcal{N}$

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_{\pm}^{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_{\pm}^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)
- Since $\mathcal{M}$ is a $\mathcal{B} - \mathrm{End}_{\mathcal{B}}(\mathcal{M})$-bimodule, one can turn $\mathcal{M}$ into a $\mathcal{B}$-bimodule category in 2 different ways: $\mathcal{M}_{\pm}$ (using $\alpha_{\pm}^{\mathcal{M}}$).
- Two monoidal 2-categories: $\mathcal{B}$-**Mod**$_{\pm}$ with products $\mathcal{M}_{\pm} \boxtimes_{\mathcal{B}} \mathcal{N}$.

Relation between $\pm$ products:

There is natural equivalence $\mathcal{N}_{-} \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}_{+} \boxtimes_{\mathcal{B}} \mathcal{N}$
given by the transposition of factors $\mathcal{N} \boxtimes \mathcal{M} \to \mathcal{M} \boxtimes \mathcal{N}$,

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_\pm^{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_\pm^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)
- Since $\mathcal{M}$ is a $\mathcal{B} - \mathrm{End}_{\mathcal{B}}(\mathcal{M})$-bimodule, one can turn $\mathcal{M}$ into a $\mathcal{B}$-bimodule category in 2 different ways: $\mathcal{M}_\pm$ (using $\alpha_\pm^{\mathcal{M}}$).
- Two monoidal 2-categories: $\mathcal{B}$-$\mathbf{Mod}_\pm$ with products $\mathcal{M}_\pm \boxtimes_{\mathcal{B}} \mathcal{N}$.

Relation between $\pm$ products:

There is natural equivalence $\mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N}$
given by the transposition of factors $\mathcal{N} \boxtimes \mathcal{M} \to \mathcal{M} \boxtimes \mathcal{N}$,
so that $\mathcal{B}$-$\mathbf{Mod}_- \simeq \mathcal{B}$-$\mathbf{Mod}_+^{\mathbf{op}}$

Let $\mathcal{B}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$.
Let $\mathcal{M}$ be a $\mathcal{B}$-module category, i.e., there is $\otimes : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$
$\mathcal{B}$-**Mod** := the 2-category of $\mathcal{B}$-module categories

Two tensor products on $\mathcal{B}$-**Mod**

- There are two tensor functors $\alpha_{\pm}^{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ (α-inductions of Böckenhauer-Evans-Kawahigashi): $\alpha_{\pm}^{\mathcal{M}}(X) = X \otimes ?$ with the $\mathcal{B}$-module structure given by $c_{X,Y}$ (resp. $c_{Y,X}^{-1}$)
- Since $\mathcal{M}$ is a $\mathcal{B} - \mathrm{End}_{\mathcal{B}}(\mathcal{M})$-bimodule, one can turn $\mathcal{M}$ into a $\mathcal{B}$-bimodule category in 2 different ways: $\mathcal{M}_{\pm}$ (using $\alpha_{\pm}^{\mathcal{M}}$).
- Two monoidal 2-categories: $\mathcal{B}$-$\mathbf{Mod}_{\pm}$ with products $\mathcal{M}_{\pm} \boxtimes_{\mathcal{B}} \mathcal{N}$.

Relation between $\pm$ products:

There is natural equivalence $\mathcal{N}_{-} \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}_{+} \boxtimes_{\mathcal{B}} \mathcal{N}$
given by the transposition of factors $\mathcal{N} \boxtimes \mathcal{M} \to \mathcal{M} \boxtimes \mathcal{N}$,
so that $\mathcal{B}$-$\mathbf{Mod}_{-} \simeq \mathcal{B}$-$\mathbf{Mod}_{+}^{\mathbf{op}}$

A *braided* $\mathcal{B}$-module category [Brochier, Ben-Zvi - Brochier - Jordan]

is a $\mathcal{B}$-module category $\mathcal{M}$ equipped with a collection of isomorphisms $\sigma^{\mathcal{M}}_{X,M} : X * M \to X * M$ (*module braiding*) natural in $X \in \mathcal{B},\ M \in \mathcal{M}$ with $\sigma_{\mathbf{1},M} = 1_M$ and such that the diagrams

$$\begin{array}{ccc}
X*(Y*M) & \xrightarrow{\sigma^{\mathcal{M}}_{X,Y*M}} & X*(Y*M) \\
\downarrow{\scriptstyle m_{X,Y,M}} & & \downarrow{\scriptstyle m_{X,Y,M}} \\
(X\otimes Y)*M & & (X\otimes Y)*M \\
\downarrow{\scriptstyle c_{X,Y}} & & \downarrow{\scriptstyle c^{-1}_{Y,X}} \\
(Y\otimes X)*M & & (Y\otimes X)*M \\
\downarrow{\scriptstyle m^{-1}_{Y,X,M}} & & \downarrow{\scriptstyle m^{-1}_{Y,X,M}} \\
Y*(X*M) & \xrightarrow{\sigma^{\mathcal{M}}_{X,M}} & Y*(X*M)
\end{array}$$

$$\begin{array}{ccccc}
(X\otimes Y)*M & & \xrightarrow{\sigma^{\mathcal{M}}_{X\otimes Y,M}} & & (X\otimes Y)*M \\
\downarrow{\scriptstyle c^{-1}_{Y,X}} & & & & \downarrow{\scriptstyle c_{X,Y}} \\
(Y\otimes X)*M & & & & (Y\otimes X)*M \\
\downarrow{\scriptstyle m^{-1}_{Y,X,M}} & & & & \downarrow{\scriptstyle m^{-1}_{Y,X,M}} \\
Y*(X*M) & \xrightarrow{\sigma^{\mathcal{M}}_{X,M}} & Y*(X*M) & \xrightarrow{\sigma^{\mathcal{M}}_{Y,X*M}} & Y*(X*M)
\end{array}$$

commute for all $X, Y \in \mathcal{B}$ and $M \in \mathcal{M}$.

A *braided* $\mathcal{B}$-module category [Brochier, Ben-Zvi - Brochier - Jordan]

is a $\mathcal{B}$-module category $\mathcal{M}$ equipped with a collection of isomorphisms $\sigma^{\mathcal{M}}_{X,M} : X * M \to X * M$ (*module braiding*) natural in $X \in \mathcal{B},\ M \in \mathcal{M}$ with $\sigma_{\mathbf{1},M} = 1_M$ and such that the diagrams

$$\begin{CD}
X*(Y*M) @>{\sigma^{\mathcal{M}}_{X,Y*M}}>> X*(Y*M)\\
@V{m_{X,Y,M}}VV @VV{m_{X,Y,M}}V\\
(X\otimes Y)*M @. (X\otimes Y)*M\\
@V{c_{X,Y}}VV @VV{c^{-1}_{Y,X}}V\\
(Y\otimes X)*M @. (Y\otimes X)*M\\
@V{m^{-1}_{Y,X,M}}VV @VV{m^{-1}_{Y,X,M}}V\\
Y*(X*M) @>{\sigma^{\mathcal{M}}_{X,M}}>> Y*(X*M)
\end{CD}$$

$$\begin{array}{ccccc}
(X\otimes Y)*M & & \xrightarrow{\sigma^{\mathcal{M}}_{X\otimes Y,M}} & & (X\otimes Y)*M\\
\downarrow{\scriptstyle c^{-1}_{Y,X}} & & & & \downarrow{\scriptstyle c_{X,Y}}\\
(Y\otimes X)*M & & & & (Y\otimes X)*M\\
\downarrow{\scriptstyle m^{-1}_{Y,X,M}} & & & & \downarrow{\scriptstyle m^{-1}_{Y,X,M}}\\
Y*(X*M) & & & & Y*(X*M)\\
& \searrow{\scriptstyle \sigma^{\mathcal{M}}_{X,M}} & & \nearrow{\scriptstyle \sigma^{\mathcal{M}}_{Y,X*M}} & \\
& & Y*(X*M) & &
\end{array}$$

commute for all $X, Y \in \mathcal{B}$ and $M \in \mathcal{M}$.

$\mathcal{B}$-module braided functors are required to respect module braiding.

Interpretation of module braidings

Interpretation of module braidings

Terminology justification

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$ It gives a $\mathcal{B}$-bimodule equivalence $\boxed{\mathcal{M}_+ \xrightarrow{\sim} \mathcal{M}_-.}$

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$ It gives a $\mathcal{B}$-bimodule equivalence $\boxed{\mathcal{M}_+ \xrightarrow{\sim} \mathcal{M}_-.}$

Tensor product of braided module categories

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$ It gives a $\mathcal{B}$-bimodule equivalence $\boxed{\mathcal{M}_+ \xrightarrow{\sim} \mathcal{M}_-.}$

Tensor product of braided module categories

$(\mathcal{M}, \sigma^{\mathcal{M}}) \boxtimes_{\mathcal{B}} (\mathcal{N}, \sigma^{\mathcal{N}}) := (\mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N}, \sigma^{\mathcal{M}} \boxtimes_{\mathcal{B}} \sigma^{\mathcal{N}})$.

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$ It gives a $\mathcal{B}$-bimodule equivalence $\boxed{\mathcal{M}_+ \xrightarrow{\sim} \mathcal{M}_-.}$

Tensor product of braided module categories

$(\mathcal{M}, \sigma^{\mathcal{M}}) \boxtimes_{\mathcal{B}} (\mathcal{N}, \sigma^{\mathcal{N}}) := (\mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N}, \sigma^{\mathcal{M}} \boxtimes_{\mathcal{B}} \sigma^{\mathcal{N}})$.
The unit object is the regular $\mathcal{B}$ with $\sigma^{\mathcal{B}}_{X,Y} = c_{Y,X} \circ c_{X,Y}$.

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$ It gives a $\mathcal{B}$-bimodule equivalence $\boxed{\mathcal{M}_+ \xrightarrow{\sim} \mathcal{M}_-.}$

Tensor product of braided module categories

$(\mathcal{M}, \sigma^{\mathcal{M}}) \boxtimes_{\mathcal{B}} (\mathcal{N}, \sigma^{\mathcal{N}}) := (\mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N}, \sigma^{\mathcal{M}} \boxtimes_{\mathcal{B}} \sigma^{\mathcal{N}})$.
The unit object is the regular $\mathcal{B}$ with $\sigma^{\mathcal{B}}_{X,Y} = c_{Y,X} \circ c_{X,Y}$.

Denote $\mathcal{B}$-$\mathbf{Mod_{br}}$ the resulting monoidal 2-category.

Interpretation of module braidings

Terminology justification

A module braiding on $\mathcal{M}$ gives rise to the pure braid group representation on $\mathrm{End}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n \otimes M)$ for $X_1, \ldots, X_n \in \mathcal{B}$ and $M \in \mathcal{M}$.

A module braiding on $\mathcal{M}$ is precisely an isomorphism of tensor functors $\boxed{\alpha_+^{\mathcal{M}} \xrightarrow{\sim} \alpha_-^{\mathcal{M}}.}$ It gives a $\mathcal{B}$-bimodule equivalence $\boxed{\mathcal{M}_+ \xrightarrow{\sim} \mathcal{M}_-.}$

Tensor product of braided module categories

$(\mathcal{M}, \sigma^{\mathcal{M}}) \boxtimes_{\mathcal{B}} (\mathcal{N}, \sigma^{\mathcal{N}}) := (\mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N}, \sigma^{\mathcal{M}} \boxtimes_{\mathcal{B}} \sigma^{\mathcal{N}})$.
The unit object is the regular $\mathcal{B}$ with $\sigma^{\mathcal{B}}_{X,Y} = c_{Y,X} \circ c_{X,Y}$.

Denote $\mathcal{B}$-$\mathbf{Mod_{br}}$ the resulting monoidal 2-category.

Module braiding = central structure

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

The above $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$ equips $\mathcal{M}$ with a structure of an object in the $\mathbf{Z}(\mathcal{B}-\mathbf{Mod})$ (= the 2-center of the monoidal 2-category $\mathcal{B}$-$\mathbf{Mod}$)

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

The above $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$ equips $\mathcal{M}$ with a structure of an object in the $\mathbf{Z}(\mathcal{B}-\mathbf{Mod})$ (= the 2-center of the monoidal 2-category $\mathcal{B}$-$\mathbf{Mod}$) and vice versa.

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

The above $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$ equips $\mathcal{M}$ with a structure of an object in the $\mathbf{Z}(\mathcal{B}-\mathbf{Mod})$ (= the 2-center of the monoidal 2-category $\mathcal{B}$-$\mathbf{Mod}$) and vice versa.

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

The above $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$ equips $\mathcal{M}$ with a structure of an object in the $\mathbf{Z}(\mathcal{B}-\mathbf{Mod})$ (= the 2-center of the monoidal 2-category $\mathcal{B}$-$\mathbf{Mod}$) and vice versa.

Thus,

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

The above $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$ equips $\mathcal{M}$ with a structure of an object in the $\mathbf{Z}(\mathcal{B}-\mathbf{Mod})$ (= the 2-center of the monoidal 2-category $\mathcal{B}$-$\mathbf{Mod}$) and vice versa.

Thus, $\mathcal{B}-\mathbf{Mod}_{\mathbf{br}} \simeq \mathbf{Z}(\mathcal{B}-\mathbf{Mod})$.

Module braiding = central structure

Let $\mathcal{N} = (\mathcal{N}, \sigma^{\mathcal{N}})$ be a braided $\mathcal{B}$-module category and $\mathcal{M}$ be any $\mathcal{B}$-module category.

Let us combine previously mentioned equivalences:

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}_+ \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\text{transposition}} \mathcal{N}_- \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\text{module braiding of } \mathcal{N}} \mathcal{N}_+ \boxtimes_{\mathcal{B}} \mathcal{M}.$$

Let us denote $\mathcal{B}-\mathbf{Mod}_+$ simply $\mathcal{B}-\mathbf{Mod}$ and its tensor product $\boxtimes_{\mathcal{B}}$.

The above $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$ equips $\mathcal{M}$ with a structure of an object in the $\mathbf{Z}(\mathcal{B}-\mathbf{Mod})$ (= the 2-center of the monoidal 2-category $\mathcal{B}$-$\mathbf{Mod}$) and vice versa.

Thus, $\mathcal{B}-\mathbf{Mod}_{\mathbf{br}} \simeq \mathbf{Z}(\mathcal{B}-\mathbf{Mod})$.
In particular, $\mathcal{B}-\mathbf{Mod}_{\mathbf{br}}$ is a *braided monoidal 2-category*.

What is a braided monoidal 2-category?

What is a braided monoidal 2-category?

Defined by Kapranov-Voevodsky, Breen.

What is a braided monoidal 2-category?

Defined by Kapranov-Voevodsky, Breen.
Just like usual braided category, but equalities now become isomorphisms (natural 2-cells):

$$\begin{aligned} \beta_{\mathcal{L},\mathcal{M},\mathcal{N}} &: (\mathrm{id}_{\mathcal{M}} \boxtimes_{\mathcal{B}} B_{\mathcal{L},\mathcal{N}})(B_{\mathcal{L},\mathcal{M}} \boxtimes_{\mathcal{B}} \mathrm{id}_{\mathcal{N}}) \xrightarrow{\sim} B_{\mathcal{L},\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}}, \\ \gamma_{\mathcal{L},\mathcal{K},\mathcal{N}} &: (B_{\mathcal{L},\mathcal{N}} \boxtimes_{\mathcal{B}} \mathrm{id}_{\mathcal{K}})(\mathrm{id}_{\mathcal{L}} \boxtimes_{\mathcal{B}} B_{\mathcal{K},\mathcal{N}}) \xrightarrow{\sim} B_{\mathcal{L}\boxtimes_{\mathcal{B}}\mathcal{K},\mathcal{N}} \end{aligned}$$

for all braided $\mathcal{B}$-module categories $\mathcal{L}$, $\mathcal{K}$, $\mathcal{M}$, $\mathcal{N}$.

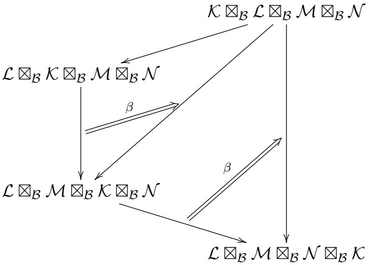
What is a braided monoidal 2-category?

Defined by Kapranov-Voevodsky, Breen.
Just like usual braided category, but equalities now become isomorphisms (natural 2-cells):

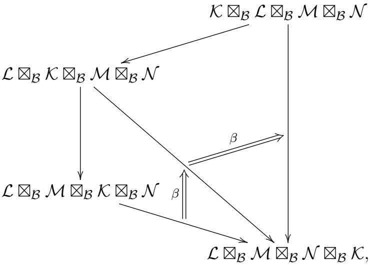
$$\begin{aligned} \beta_{\mathcal{L},\mathcal{M},\mathcal{N}} &: (\mathrm{id}_{\mathcal{M}} \boxtimes_{\mathcal{B}} B_{\mathcal{L},\mathcal{N}})(B_{\mathcal{L},\mathcal{M}} \boxtimes_{\mathcal{B}} \mathrm{id}_{\mathcal{N}}) \xrightarrow{\sim} B_{\mathcal{L},\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}}, \\ \gamma_{\mathcal{L},\mathcal{K},\mathcal{N}} &: (B_{\mathcal{L},\mathcal{N}} \boxtimes_{\mathcal{B}} \mathrm{id}_{\mathcal{K}})(\mathrm{id}_{\mathcal{L}} \boxtimes_{\mathcal{B}} B_{\mathcal{K},\mathcal{N}}) \xrightarrow{\sim} B_{\mathcal{L}\boxtimes_{\mathcal{B}}\mathcal{K},\mathcal{N}} \end{aligned}$$

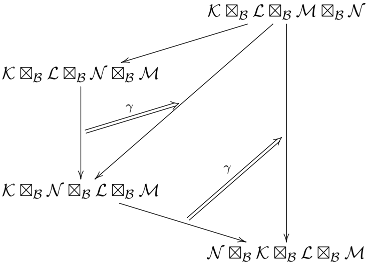
for all braided $\mathcal{B}$-module categories $\mathcal{L}$, $\mathcal{K}$, $\mathcal{M}$, $\mathcal{N}$.

These satisfy coherence of their own.

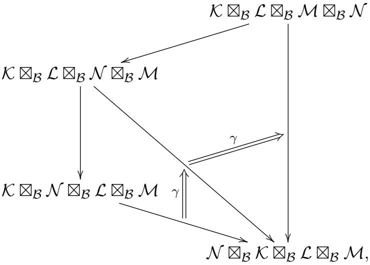

$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$
$\mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$
β
β
$\mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{N}$
$\mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{K}$

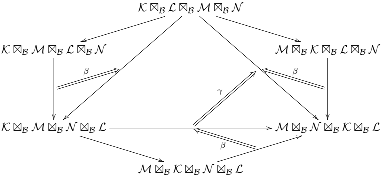
=


$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$
$\mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$
β
$\mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{N}$
β
$\mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{K}$,

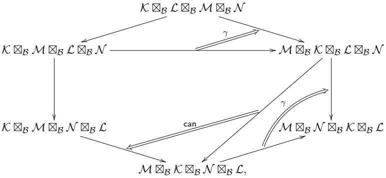

$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$
$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$
γ
γ
$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M}$
$\mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M}$

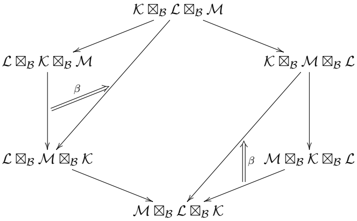
=


$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$
$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$
γ
$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M}$
γ
$\mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{L} \boxtimes_{\mathcal{B}} \mathcal{M}$,

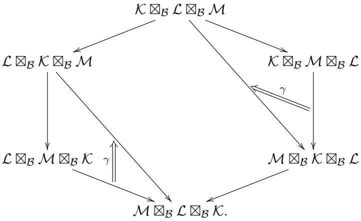

$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{N}$
$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{N}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{N}$
β
β
γ
$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{N} \boxtimes_\mathcal{B} \mathcal{L}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{N} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{L}$
β
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{N} \boxtimes_\mathcal{B} \mathcal{L}$

=


$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{N}$
γ
$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{N}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{N}$
γ
can
$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{N} \boxtimes_\mathcal{B} \mathcal{L}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{N} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{L}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{N} \boxtimes_\mathcal{B} \mathcal{L}$,


$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{M}$
$\mathcal{L} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{M}$
$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{L}$
β
$\mathcal{L} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{K}$
β
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{L}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{K}$

=


$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{M}$
$\mathcal{L} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{M}$
$\mathcal{K} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{L}$
γ
$\mathcal{L} \boxtimes_\mathcal{B} \mathcal{M} \boxtimes_\mathcal{B} \mathcal{K}$
γ
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{K} \boxtimes_\mathcal{B} \mathcal{L}$
$\mathcal{M} \boxtimes_\mathcal{B} \mathcal{L} \boxtimes_\mathcal{B} \mathcal{K}$.

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

For our purposes we will need the "pointed part" of $\mathcal{B}-\mathbf{Mod_{br}}$ consisting of braided module categories invertible w.r.t. $\boxtimes_{\mathcal{B}}$ and equivalences between them: $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathcal{B}-\mathbf{Mod}_{br})$.

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

For our purposes we will need the "pointed part" of $\mathcal{B}-\mathbf{Mod_{br}}$ consisting of braided module categories invertible w.r.t. $\boxtimes_{\mathcal{B}}$ and equivalences between them: $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathcal{B}-\mathbf{Mod}_{br})$.

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

For our purposes we will need the "pointed part" of $\mathcal{B}-\mathbf{Mod_{br}}$ consisting of braided module categories invertible w.r.t. $\boxtimes_{\mathcal{B}}$ and equivalences between them: $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathcal{B}-\mathbf{Mod}_{br})$.

If we view $\mathbf{Pic_{br}}(\mathcal{B})$ as a 3-categorical group with a single object, then the homotopy groups of the corresponding topological space are

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

For our purposes we will need the "pointed part" of $\mathcal{B}-\mathbf{Mod_{br}}$ consisting of braided module categories invertible w.r.t. $\boxtimes_{\mathcal{B}}$ and equivalences between them: $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathcal{B}-\mathbf{Mod}_{br})$.

If we view $\mathbf{Pic_{br}}(\mathcal{B})$ as a 3-categorical group with a single object, then the homotopy groups of the corresponding topological space are $\pi_1 = 1$, $\pi_2 = \mathrm{Pic}_{br}(\mathcal{B})$, $\pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$, and $\pi_4 = k^\times$.

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

For our purposes we will need the "pointed part" of $\mathcal{B}-\mathbf{Mod_{br}}$ consisting of braided module categories invertible w.r.t. $\boxtimes_{\mathcal{B}}$ and equivalences between them: $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathcal{B}-\mathbf{Mod}_{br})$.

If we view $\mathbf{Pic_{br}}(\mathcal{B})$ as a 3-categorical group with a single object, then the homotopy groups of the corresponding topological space are $\pi_1 = 1$, $\pi_2 = \mathrm{Pic}_{br}(\mathcal{B})$, $\pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$, and $\pi_4 = k^\times$.

There is an exact sequence for the underlying group $\mathrm{Pic}_{br}(\mathcal{B})$ of $\mathbf{Pic_{br}}(\mathcal{B})$:

$$0 \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathrm{Inv}(\mathcal{B}) \to \mathrm{Aut}_{\otimes}(\mathrm{id}_{\mathcal{B}}) \to \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Pic}(\mathcal{B}) \to \mathrm{Aut}_{br}(\mathcal{B}).$$

The braided 2-categorical Picard group $\mathbf{Pic_{br}}(\mathcal{B})$

For our purposes we will need the "pointed part" of $\mathcal{B}-\mathbf{Mod_{br}}$ consisting of braided module categories invertible w.r.t. $\boxtimes_{\mathcal{B}}$ and equivalences between them: $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathcal{B}-\mathbf{Mod}_{br})$.

If we view $\mathbf{Pic_{br}}(\mathcal{B})$ as a 3-categorical group with a single object, then the homotopy groups of the corresponding topological space are $\pi_1 = 1$, $\pi_2 = \mathrm{Pic}_{br}(\mathcal{B})$, $\pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$, and $\pi_4 = k^\times$.

There is an exact sequence for the underlying group $\mathrm{Pic}_{br}(\mathcal{B})$ of $\mathbf{Pic_{br}}(\mathcal{B})$:

$$0 \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathrm{Inv}(\mathcal{B}) \to \mathrm{Aut}_{\otimes}(\mathrm{id}_{\mathcal{B}}) \to \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Pic}(\mathcal{B}) \to \mathrm{Aut}_{br}(\mathcal{B}).$$

Here Inv() denotes the group of invertible objects, $\mathrm{Pic}(\mathcal{B})$ is the usual Picard group of $\mathcal{B}$.

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

$\mathrm{Pic}_{br}(\mathcal{B})$ (the 1-categorical truncation of $\mathbf{Pic_{br}}(\mathcal{B})$) is a braided categorical group.

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

$\mathrm{Pic}_{br}(\mathcal{B})$ (the 1-categorical truncation of $\mathbf{Pic_{br}}(\mathcal{B})$) is a braided categorical group.
So there is a canonical quadratic form

$$Q_{\mathcal{B}} : \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$$

by [Joyal-Street].

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

$\mathrm{Pic}_{br}(\mathcal{B})$ (the 1-categorical truncation of $\mathbf{Pic_{br}}(\mathcal{B})$) is a braided categorical group.
So there is a canonical quadratic form

$$Q_{\mathcal{B}} : \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$$

by [Joyal-Street].
This comes from $\pi_2 \times \pi_2 \to \pi_3$.

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

$\mathrm{Pic}_{br}(\mathcal{B})$ (the 1-categorical truncation of $\mathbf{Pic_{br}}(\mathcal{B})$) is a braided categorical group.
So there is a canonical quadratic form

$$Q_{\mathcal{B}} : \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$$

by [Joyal-Street].
This comes from $\pi_2 \times \pi_2 \to \pi_3$.

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

$\mathrm{Pic}_{br}(\mathcal{B})$ (the 1-categorical truncation of $\mathbf{Pic_{br}}(\mathcal{B})$) is a braided categorical group.
So there is a canonical quadratic form

$$Q_{\mathcal{B}} : \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$$

by [Joyal-Street].
This comes from $\pi_2 \times \pi_2 \to \pi_3$.

There is a well-defined bilinear map

$$P_{\mathcal{B}} : \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \times \mathrm{Pic}_{br}(\mathcal{B}) \to k^\times$$

given by $P_{\mathcal{B}}(Z, \mathcal{M}) = \sigma_{Z,X} \in \mathrm{Aut}(Z \otimes X) = k^\times$, $X \in \mathcal{M}$.

Whitehead products $\pi_k \times \pi_l \to \pi_{k+l-1}$

$\mathrm{Pic}_{br}(\mathcal{B})$ (the 1-categorical truncation of $\mathbf{Pic_{br}}(\mathcal{B})$) is a braided categorical group.
So there is a canonical quadratic form

$$Q_{\mathcal{B}} : \mathrm{Pic}_{br}(\mathcal{B}) \to \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$$

by [Joyal-Street].
This comes from $\pi_2 \times \pi_2 \to \pi_3$.

There is a well-defined bilinear map

$$P_{\mathcal{B}} : \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \times \mathrm{Pic}_{br}(\mathcal{B}) \to k^\times$$

given by $P_{\mathcal{B}}(Z, \mathcal{M}) = \sigma_{Z,X} \in \mathrm{Aut}(Z \otimes X) = k^\times$, $X \in \mathcal{M}$.
This is $\pi_3 \times \pi_2 \to \pi_4$.

Outline

From extensions to braided monoidal 2-functors and back

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \text{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \mathrm{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \mathrm{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

Furthermore, the tensor products $\otimes_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ gives rise to $\mathcal{B}$-module equivalences $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$.

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \mathrm{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

Furthermore, the tensor products $\otimes_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ gives rise to $\mathcal{B}$-module equivalences $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$.

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \mathrm{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

Furthermore, the tensor products $\otimes_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ gives rise to $\mathcal{B}$-module equivalences $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$.

This gives a (usual) monoidal functor

$$A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x.$$

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \mathrm{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

Furthermore, the tensor products $\otimes_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ gives rise to $\mathcal{B}$-module equivalences $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$.

This gives a (usual) monoidal functor

$$A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x.$$

From extensions to braided monoidal 2-functors and back

Let A be a finite Abelian group. Let $\mathcal{B}$ be a braided fusion category with braiding c.

Given a braided extension

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B},$$

we have $\mathcal{C}_x \in \mathrm{Pic}_{br}(\mathcal{B})$, $x \in X$ with the module braiding given by $\sigma_{X,V} = c_{X,V} c_{V,X}$, $V \in \mathcal{B}$, $X \in \mathcal{C}_x$.

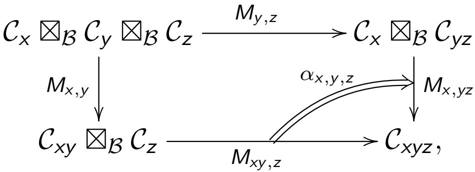
Furthermore, the tensor products $\otimes_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ gives rise to $\mathcal{B}$-module equivalences $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$.

This gives a (usual) monoidal functor

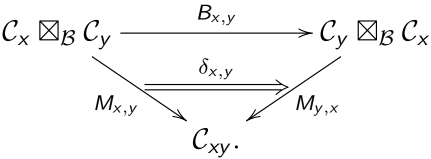
$$A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x.$$

Namely, the associativity and braiding constraints of $\mathcal{C}$ give rise to natural 2-cells involving $M_{x,y}$, $x, y \in A$:

Namely, the associativity and braiding constraints of $\mathcal{C}$ give rise to natural 2-cells involving $M_{x,y}$, $x, y \in A$:



and



Here $B_{x,y}$ is the braiding in $\mathbf{Pic_{br}}(\mathcal{B})$.

The moral: Structure morphisms in $\mathcal{C} \longleftrightarrow$ structure 2-cells in $\mathbf{Pic_{br}}(\mathcal{B})$.

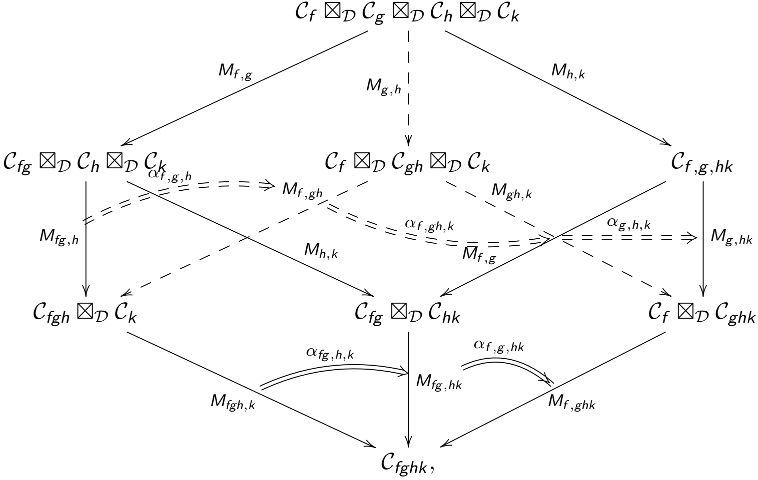
Consequently, the pentagon (for the associativity of $\mathcal{C}$) and two hexagons (for the braiding of $\mathcal{C}$) diagrams $\longleftrightarrow$ commuting polytopes in $\mathbf{Pic_{br}}(\mathcal{B})$.

Consequently, the pentagon (for the associativity of $\mathcal{C}$) and two hexagons (for the braiding of $\mathcal{C}$) diagrams $\longleftrightarrow$ commuting polytopes in $\mathbf{Pic_{br}}(\mathcal{B})$.

Namely, the pentagon becomes a cube:

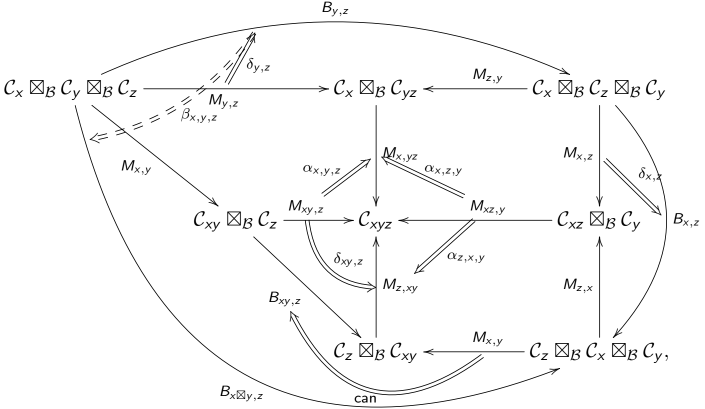
Consequently, the pentagon (for the associativity of $\mathcal{C}$) and two hexagons (for the braiding of $\mathcal{C}$) diagrams $\longleftrightarrow$ commuting polytopes in $\mathbf{Pic_{br}}(\mathcal{B})$.

Namely, the pentagon becomes a cube:

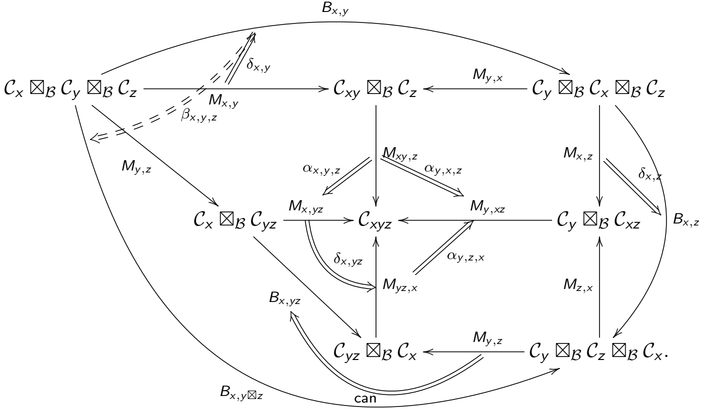


and hexagons become octahedra:

and hexagons become octahedra:



and


$B_{x,y}$
$\delta_{x,y}$
$\mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \boxtimes_{\mathcal{B}} \mathcal{C}_z$
$M_{x,y}$
$\mathcal{C}_{xy} \boxtimes_{\mathcal{B}} \mathcal{C}_z$
$M_{y,x}$
$\mathcal{C}_y \boxtimes_{\mathcal{B}} \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_z$
$\beta_{x,y,z}$
$M_{y,z}$
$M_{xy,z}$
$\alpha_{x,y,z}$
$\alpha_{y,x,z}$
$M_{x,z}$
$\delta_{x,z}$
$B_{x,z}$
$\mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_{yz}$
$M_{x,yz}$
$\mathcal{C}_{xyz}$
$M_{y,xz}$
$\mathcal{C}_y \boxtimes_{\mathcal{B}} \mathcal{C}_{xz}$
$\delta_{x,yz}$
$\alpha_{y,z,x}$
$M_{yz,x}$
$M_{z,x}$
$B_{x,yz}$
$\mathcal{C}_{yz} \boxtimes_{\mathcal{B}} \mathcal{C}_x$
$M_{y,z}$
$\mathcal{C}_y \boxtimes_{\mathcal{B}} \mathcal{C}_z \boxtimes_{\mathcal{B}} \mathcal{C}_x.$
$B_{x,y\boxtimes z}$
can

A functor $A \to \mathbf{Pic_{br}}(\mathcal{B})$ with these structures (associativity and braiding cells α and δ) such that the above polytopes commute is a *braided monoidal 2-functor*. So we went from extensions to functors.

A functor $A \to \mathbf{Pic_{br}}(\mathcal{B})$ with these structures (associativity and braiding cells α and δ) such that the above polytopes commute is a *braided monoidal 2-functor*. So we went from extensions to functors.

Conversely, given a braided monoidal 2-functor $A \to \mathbf{Pic_{br}}(\mathcal{B}), x \mapsto \mathcal{C}_x$, i.e., $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}\, (x, y \in A)$ and cells α and δ such that the polytopes commute we form a fusion category

$$\mathcal{C} = \oplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B}$$

and equip it with the tensor product $\otimes_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$ (coming from $M_{x,y}$) and associativity and braiding constraints (coming from α and δ) and get a braided fusion category.

A functor $A \to \mathbf{Pic_{br}}(\mathcal{B})$ with these structures (associativity and braiding cells α and δ) such that the above polytopes commute is a *braided monoidal 2-functor*. So we went from extensions to functors.

Conversely, given a braided monoidal 2-functor $A \to \mathbf{Pic_{br}}(\mathcal{B}), x \mapsto \mathcal{C}_x$, i.e., $M_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}\,(x, y \in A)$ and cells α and δ such that the polytopes commute we form a fusion category

$$\mathcal{C} = \oplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_1 = \mathcal{B}$$

and equip it with the tensor product $\otimes_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$ (coming from $M_{x,y}$) and associativity and braiding constraints (coming from α and δ) and get a braided fusion category.

Main theorem

$\{$ Groupoid of braided A-extensions of $\mathcal{B}$ $\} \simeq \{$ groupoid of *braided* monoidal 2-functors $\boxed{A \to \mathbf{Pic_{br}}(\mathcal{B})}\}$

Understanding obstructions

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic_{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A, N) = H^{n+1}(K(A, 2), N).$$

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic_{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A,\, N) = H^{n+1}(K(A,\, 2),\, N).$$

- $H^2_{ab}(A, N) = \mathrm{Ext}_{\mathbb{Z}}(A, N)$: symmetric 2-cocycles $\longrightarrow$ abelian groups,
- $H^3_{ab}(A, N) = \mathrm{Quad}(A,\, N)$: abelian 3-cocycles $= (\omega : A^3 \to N,\, c : A^2 \to N)$ satisfying pentagon $+$ 2 hexagons $\longrightarrow$ braided categorical groups ,
- $H^4_{ab}(A, N) =$ triples $(a : A^4 \to N,\, \beta, \gamma : A^3 \to N)$ satisfying certain coherence conditions (cf. polytopes in the definition of a braided monoidal 2-category) $\longrightarrow$ braided 2-categorical groups.

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic_{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A,\, N) = H^{n+1}(K(A,\, 2),\, N).$$

- $H^2_{ab}(A, N) = \mathrm{Ext}_{\mathbb{Z}}(A, N)$: symmetric 2-cocycles $\longrightarrow$ abelian groups,
- $H^3_{ab}(A, N) = \mathrm{Quad}(A,\, N)$: abelian 3-cocycles $= (\omega : A^3 \to N,\, c : A^2 \to N)$ satisfying pentagon + 2 hexagons $\longrightarrow$ braided categorical groups ,
- $H^4_{ab}(A, N) =$ triples $(a : A^4 \to N,\, \beta, \gamma : A^3 \to N)$ satisfying certain coherence conditions (cf. polytopes in the definition of a braided monoidal 2-category) $\longrightarrow$ braided 2-categorical groups.

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic_{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A,\, N) = H^{n+1}(K(A,\, 2),\, N).$$

- $H^2_{ab}(A, N) = \mathrm{Ext}_{\mathbb{Z}}(A, N)$: symmetric 2-cocycles $\longrightarrow$ abelian groups,
- $H^3_{ab}(A, N) = \mathrm{Quad}(A,\, N)$: abelian 3-cocycles $= (\omega : A^3 \to N,\, c : A^2 \to N)$ satisfying pentagon + 2 hexagons $\longrightarrow$ braided categorical groups ,
- $H^4_{ab}(A, N) =$ triples $(a : A^4 \to N,\, \beta, \gamma : A^3 \to N)$ satisfying certain coherence conditions (cf. polytopes in the definition of a braided monoidal 2-category) $\longrightarrow$ braided 2-categorical groups.

A (usual) braided monoidal functor $M : A \to \mathrm{Pic}_{br}(\mathcal{B})$ gives rise to a braided monoidal 2-functor (i.e., to a braided A-extension of $\mathcal{B}$) $\Longleftrightarrow$

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic_{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A,\, N) = H^{n+1}(K(A,\, 2),\, N).$$

- $H^2_{ab}(A, N) = \mathrm{Ext}_{\mathbb{Z}}(A, N)$: symmetric 2-cocycles $\longrightarrow$ abelian groups,
- $H^3_{ab}(A, N) = \mathrm{Quad}(A,\, N)$: abelian 3-cocycles $= (\omega : A^3 \to N,\, c : A^2 \to N)$ satisfying pentagon + 2 hexagons $\longrightarrow$ braided categorical groups ,
- $H^4_{ab}(A, N) =$ triples $(a : A^4 \to N,\, \beta, \gamma : A^3 \to N)$ satisfying certain coherence conditions (cf. polytopes in the definition of a braided monoidal 2-category) $\longrightarrow$ braided 2-categorical groups.

A (usual) braided monoidal functor $M : A \to \mathrm{Pic}_{br}(\mathcal{B})$ gives rise to a braided monoidal 2-functor (i.e., to a braided A-extension of $\mathcal{B}$) $\Longleftrightarrow$ an obstruction $\boxed{o_4(M) \in H^4_{ab}(A,\, k^\times)}$ (given by the cube + 2 octahedra above) vanishes.

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic_{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A,\, N) = H^{n+1}(K(A,\,2),\, N).$$

- $H^2_{ab}(A,N) = \mathrm{Ext}_{\mathbb{Z}}(A,N)$: symmetric 2-cocycles $\longrightarrow$ abelian groups,
- $H^3_{ab}(A,N) = \mathrm{Quad}(A,\,N)$: abelian 3-cocycles $= (\omega : A^3 \to N,\ c : A^2 \to N)$ satisfying pentagon + 2 hexagons $\longrightarrow$ braided categorical groups ,
- $H^4_{ab}(A,N) =$ triples $(a : A^4 \to N,\ \beta,\gamma : A^3 \to N)$ satisfying certain coherence conditions (cf. polytopes in the definition of a braided monoidal 2-category) $\longrightarrow$ braided 2-categorical groups.

A (usual) braided monoidal functor $M : A \to \mathrm{Pic}_{br}(\mathcal{B})$ gives rise to a braided monoidal 2-functor (i.e., to a braided A-extension of $\mathcal{B}$) $\Longleftrightarrow$ an obstruction $\boxed{o_4(M) \in H^4_{ab}(A,\ k^\times)}$ (given by the cube + 2 octahedra above) vanishes.

In this case 2-functors are parameterized by an $H^3_{ab}(A,\ k^\times)$-torsor.

Understanding obstructions

Braided monoidal 2-functors $A \to \mathbf{Pic}_{\mathbf{br}}(\mathcal{B})$ can be understood using the Eilenberg-MacLane cohomology of abelian groups

$$H^n_{ab}(A,\, N) = H^{n+1}(K(A,\, 2),\, N).$$

- $H^2_{ab}(A, N) = \mathrm{Ext}_{\mathbb{Z}}(A, N)$: symmetric 2-cocycles $\longrightarrow$ abelian groups,
- $H^3_{ab}(A, N) = \mathrm{Quad}(A,\, N)$: abelian 3-cocycles $= (\omega : A^3 \to N,\, c : A^2 \to N)$ satisfying pentagon + 2 hexagons $\longrightarrow$ braided categorical groups ,
- $H^4_{ab}(A, N) =$ triples $(a : A^4 \to N,\, \beta, \gamma : A^3 \to N)$ satisfying certain coherence conditions (cf. polytopes in the definition of a braided monoidal 2-category) $\longrightarrow$ braided 2-categorical groups.

A (usual) braided monoidal functor $M : A \to \mathrm{Pic}_{br}(\mathcal{B})$ gives rise to a braided monoidal 2-functor (i.e., to a braided A-extension of $\mathcal{B}$) $\Longleftrightarrow$ an obstruction $\boxed{o_4(M) \in H^4_{ab}(A,\, k^\times)}$ (given by the cube + 2 octahedra above) vanishes.

In this case 2-functors are parameterized by an $H^3_{ab}(A,\, k^\times)$-torsor.

The Pontryagin-Whitehead quadratic function

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \text{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A,\, \text{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A,\, \text{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.
(So the tensor product $V_x \otimes U_y$ is replaced by $L_{x,y} \otimes V_x \otimes U_y$ etc. This was called "zesting" in the literature).

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.
(So the tensor product $V_x \otimes U_y$ is replaced by $L_{x,y} \otimes V_x \otimes U_y$ etc. This was called "zesting" in the literature).

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.
(So the tensor product $V_x \otimes U_y$ is replaced by $L_{x,y} \otimes V_x \otimes U_y$ etc. This was called "zesting" in the literature).

$$o_4(L \circ M) = o_4(M) pw_M(L) \quad \text{in } H^4_{ab}(A,\, k^\times),$$

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.
(So the tensor product $V_x \otimes U_y$ is replaced by $L_{x,y} \otimes V_x \otimes U_y$ etc. This was called "zesting" in the literature).

$$o_4(L \circ M) = o_4(M) pw_M(L) \quad \text{in } H^4_{ab}(A, k^\times),$$

where $pw_M(L) = (a(L), \beta_M(L), \gamma_M(L))$ with

The Pontryagin-Whitehead quadratic function

Braided monoidal functors $M : A \to \mathrm{Pic}_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$ form a torsor over $H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

Given $L \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$ compose $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \to \mathcal{C}_{xy}$ with the tensor multiplication by $L_{x,y}$. Denote the new functor $L \circ M$.
(So the tensor product $V_x \otimes U_y$ is replaced by $L_{x,y} \otimes V_x \otimes U_y$ etc. This was called "zesting" in the literature).

$$o_4(L \circ M) = o_4(M) pw_M(L) \quad \text{in } H^4_{ab}(A, k^\times),$$

where $pw_M(L) = (a(L), \beta_M(L), \gamma_M(L))$ with

$$\begin{aligned} a(L) &: A^4 \to k^\times, \\ \beta_M(L) &: A^3 \to k^\times, \\ \gamma_M(L) &: A^3 \to k^\times \end{aligned}$$

defined as follows:

$pw_M(L) = (a(L), \beta_M(L), \gamma_M(L)) \in H^4_{ab}(A, k^\times)$

Here $L = \{L_{x,y}\} \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

$pw_M(L) = (a(L), \beta_M(L), \gamma_M(L)) \in H^4_{ab}(A, k^\times)$

Here $L = \{L_{x,y}\} \in H^2_{ab}(A, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

$a(L) \in H^4(A, k^\times)$ comes from the self braiding

$$\mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathbb{Z}_2 \subset k^\times : Z \mapsto c_{Z,Z}$$

(in our case it is a homomorphism) composed with the cup product square:

$pw_M(L) = (a(L),\, \beta_M(L), \gamma_M(L)) \in H^4_{ab}(A,\, k^\times)$

Here $L = \{L_{x,y}\} \in H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

$a(L) \in H^4(A,\, k^\times)$ comes from the self braiding

$$\mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathbb{Z}_2 \subset k^\times : Z \mapsto c_{Z,Z}$$

(in our case it is a homomorphism) composed with the cup product square:

$$H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))) \to H^2_{ab}(A,\, \mathbb{Z}_2) \xrightarrow{\cup^2} H^4(A,\, \mathbb{Z}_2) \to H^4(A,\, k^\times),$$

$pw_M(L) = (a(L),\, \beta_M(L), \gamma_M(L)) \in H^4_{ab}(A,\, k^\times)$

Here $L = \{L_{x,y}\} \in H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

$a(L) \in H^4(A,\, k^\times)$ comes from the self braiding

$$\mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathbb{Z}_2 \subset k^\times : Z \mapsto c_{Z,Z}$$

(in our case it is a homomorphism) composed with the cup product square:

$$H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))) \to H^2_{ab}(A,\, \mathbb{Z}_2) \xrightarrow{\cup^2} H^4(A,\, \mathbb{Z}_2) \to H^4(A,\, k^\times),$$

$\beta_M(L), \gamma_M(L) : A^3 \to k^\times$ are defined using the map
$P_{\mathcal{B}} : \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \times \mathrm{Pic}_{br}(\mathcal{B}) \to k^\times$ (i.e., $\pi_3 \times \pi_2 \to \pi_4$) by

$pw_M(L) = (a(L),\, \beta_M(L), \gamma_M(L)) \in H^4_{ab}(A,\, k^\times)$

Here $L = \{L_{x,y}\} \in H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))$.

$a(L) \in H^4(A,\, k^\times)$ comes from the self braiding

$$\mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathbb{Z}_2 \subset k^\times : Z \mapsto c_{Z,Z}$$

(in our case it is a homomorphism) composed with the cup product square:

$$H^2_{ab}(A,\, \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))) \to H^2_{ab}(A,\, \mathbb{Z}_2) \xrightarrow{\cup^2} H^4(A,\, \mathbb{Z}_2) \to H^4(A,\, k^\times),$$

$\beta_M(L), \gamma_M(L) : A^3 \to k^\times$ are defined using the map $P_{\mathcal{B}} : \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B})) \times \mathrm{Pic}_{br}(\mathcal{B}) \to k^\times$ (i.e., $\pi_3 \times \pi_2 \to \pi_4$) by

$$\begin{aligned} \beta_M(L)(x,y,z) &= P_{\mathcal{B}}(L_{y,z},\, \mathcal{C}_x) \\ \gamma_M(L)(x,y,z) &= P_{\mathcal{B}}(L_{x,y},\, \mathcal{C}_z). \end{aligned}$$

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible).

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Example: $\mathcal{B} = \mathrm{sVec}$

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Example: $\mathcal{B} = \mathrm{sVec}$

$\mathrm{Pic}_{br}(\mathrm{sVec}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Example: $\mathcal{B} = \mathrm{sVec}$

$\mathrm{Pic}_{br}(\mathrm{sVec}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Q_{sVec} takes values $\{I, I, I, \Pi\}$, where $\mathrm{Inv}(\mathrm{sVec}) = \{I, \Pi\}$.

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Example: $\mathcal{B} = \mathrm{sVec}$

$\mathrm{Pic}_{br}(\mathrm{sVec}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Q_{sVec} takes values $\{I, I, I, \Pi\}$, where $\mathrm{Inv}(\mathrm{sVec}) = \{I, \Pi\}$.

Example: $\mathcal{B}$ is Tannakian

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Example: $\mathcal{B} = \mathrm{sVec}$

$\mathrm{Pic}_{br}(\mathrm{sVec}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Q_{sVec} takes values $\{I, I, I, \Pi\}$, where $\mathrm{Inv}(\mathrm{sVec}) = \{I, \Pi\}$.

Example: $\mathcal{B}$ is Tannakian

$\mathrm{Pic}_{br}(\mathrm{Rep}(G)) \cong H^2(G, k^\times) \times Z(G)$ with

Computing $\mathbf{Pic_{br}}(\mathcal{B})$

The braided categorical group structure is determined by the canonical quadratic form

$$Q_{\mathcal{B}} : \pi_2 = \mathrm{Pic}_{br}(\mathcal{B}) \longrightarrow \pi_3 = \mathrm{Inv}(\mathcal{Z}_{sym}(\mathcal{B}))).$$

Example: $\mathcal{B}$ is non-degenerate

Then $\mathbf{Pic_{br}}(\mathcal{B})$ is trivial (i.e., contactible). Only trivial braided extensions (tensoring with a pointed category): $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{C}(A, q)$.

Example: $\mathcal{B} = \mathrm{sVec}$

$\mathrm{Pic}_{br}(\mathrm{sVec}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Q_{sVec} takes values $\{I, I, I, \Pi\}$, where $\mathrm{Inv}(\mathrm{sVec}) = \{I, \Pi\}$.

Example: $\mathcal{B}$ is Tannakian

$\mathrm{Pic}_{br}(\mathrm{Rep}(G)) \cong H^2(G, k^\times) \times Z(G)$ with

$$Q_{\mathrm{Rep}(G)} : H^2(G, k^\times) \times Z(G) \to \widehat{G}, \qquad Q_{\mathrm{Rep}(G)}(\mu, z) = \frac{\mu(z, -)}{\mu(-, z)}.$$

Thanks for listening!