UNIQUENESS AND EXISTENCE RESULTS

VIA

MORSE INDEX

FOR

LANE-EMDEN PROBLEMS

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Physical Geometrical and Analytical Aspects of Mean Field Systems of Liouville Type
Banff International Research Station, April 1-6, 2018

The Lane Emden problem

We consider the classical Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (*)$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$.

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where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$.

We will see as suitable rescalings of solutions u_p of $(*)$ converge to a solution of (RL) or (SL) as $p \to +\infty$, where

$$(RL) \quad \begin{cases} -\Delta \mathcal{U} = e^{\mathcal{U}} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{\mathcal{U}} dx < +\infty \end{cases} \qquad (SL) \quad \begin{cases} -\Delta \mathcal{V} = e^{\mathcal{V}} - 4\pi\eta\delta_0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{\mathcal{V}} dx < +\infty \end{cases}$$

Finite energy solutions

Theorem [Ren - Wei, Trans. Amer. Math. Soc. 1994]

For any family $(u_p)_p$ of nontrivial solutions

$$\liminf_{p\to+\infty} p\int_\Omega |\nabla u_p|^2 dx \geq 8\pi e.$$

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Finite energy condition (which is NOT known to hold in general domains)

We will assume that given $p_0 > 1$ there exists $C = C(p_0, \Omega) > 0$ such that for any $p \geq p_0$

$$p \int_\Omega |\nabla u_p|^2 dx \leq C. \qquad (F)$$

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FIRST EXAMPLES

- Least energy positive solutions: $p\int_\Omega |\nabla u_p|^2 dx \to 8\pi e$ as $p \to +\infty$
  [Ren - Wei, Trans. Amer. Math. Soc. 1994]

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First Examples

- Least energy positive solutions: $p\int_\Omega |\nabla u_p|^2 dx \to 8\pi e$ as $p \to +\infty$
  [Ren - Wei, Trans. Amer. Math. Soc. 1994]
- Least energy sign-changing solutions: $p\int_\Omega |\nabla u_p|^2 dx \to 16\pi e$ as $p \to +\infty$
  [Castro - Cossio - Neuberger, Rocky Mount. J. of Math 1997]

No vanishing - No blow-up

Lemma [Ren - Wei, Trans. Amer. Math. Soc. 1994]

For any family $(u_p)_p$ of nontrivial solutions satisfying (F)

- $\liminf_{p\to+\infty} \|u_p\|_{L^\infty(\Omega)} \geq 1$;
- $\|u_p\|_{L^\infty(\Omega)} \leq C$, for some C independent of p.

A first bubble

Let us assume w.l.o.g. that

$$\|u_p\|_{L^\infty(\Omega)} = \|u_p^+\|_{L^\infty(\Omega)}$$

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Theorem [Adimurthi - Grossi, Proc. Amer. Math. Soc. 2004]

Let $(u_p)_p$ be a family of nontrivial solutions to $(*)$, satisfying (F).
Let $x_p^+ \in \Omega$ be such that $u_p(x_p^+) = \|u_p\|_{L^\infty(\Omega)}$ and let us define the scaling parameter

$$\varepsilon_p^+ = \frac{1}{\sqrt{p\,(u_p(x_p^+))^{p-1}}}.$$

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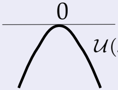
$$\varepsilon_p^+ = \frac{1}{\sqrt{p\,(u_p(x_p^+))^{p-1}}}.$$

Then, up to a subsequence, the following scaled function about x_p^+ verifies

$$w_p^+(x) = p\,\frac{u_p(x_p^+ + \varepsilon_p^+ x) - u_p(x_p^+)}{u_p(x_p^+)} \underset{p\to+\infty}{\longrightarrow} \mathcal{U}(x) \qquad \text{in } C^1_{loc}(\mathbb{R}^2)$$

where

$$\begin{cases} -\Delta\mathcal{U} = e^{\mathcal{U}} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{\mathcal{U}} = 8\pi & \\ \mathcal{U}(0) = 0, \mathcal{U} \le 0 & \end{cases}$$



$$\mathcal{U}(x) = 2\log\left(\frac{1}{1+\frac{1}{8}|x|^2}\right)$$

Theorem [D - Ianni - Pacella, J. Eur. Math. Soc. 2015]

Let (u_p) be positive/sign-changing solutions to $(*)$ satisfying (F).
Then there exists $k \in \mathbb{N} \setminus \{0\}$ and k families of points $x_{1,p} := x_p^+, x_{2,p}, \ldots, x_{k,p}$ in Ω such that, after passing to a subsequence,

$(\mathcal{P}_0^k)$ $\quad (\varepsilon_{i,p})^{-2} := p|u_p(x_{i,p})|^{p-1} \to +\infty \qquad$ (hence $|u_p(x_{i,p})| \geq 1 - \delta$)

$(\mathcal{P}_1^k)$ $\quad \lim_p \frac{|x_{i,p} - x_{j,p}|}{\varepsilon_{i,p}} = +\infty \quad$ for $i \neq j$

$(\mathcal{P}_2^k)$ $\quad w_{i,p}(x) := p \frac{u_p(x_{i,p} + \varepsilon_{i,p} x) - u_p(x_{i,p})}{u_p(x_{i,p})} \underset{p \to +\infty}{\longrightarrow} \mathcal{U}(x) \quad$ in $C^1_{loc}(\mathbb{R}^2)$

$(\mathcal{P}_3^k)$ $\quad$ there exists $C > 0$ such that:

$\min_{i=1,\ldots,k} |x - x_{i,p}|^2 \, p|u_p(x)|^{p-1} \leq C \quad$ for all $x \in \Omega$

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Moreover, given any other family of points $x_{k+1,p}$ it is impossible to extract a new sequence such that $(\mathcal{P}_0^{k+1})$, $(\mathcal{P}_1^{k+1})$, $(\mathcal{P}_2^{k+1})$ and $(\mathcal{P}_3^{k+1})$ hold.

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Last, defining the concentration set as

$$\mathcal{S} = \{\lim_{p \to +\infty} x_{i,p} \,|\, i = 1, \ldots, k\} = \{x_{1,\infty}, \ldots, x_{N,\infty}\} \subset \bar{\Omega}$$

$$\sqrt{p}\, u_p \to 0 \quad \text{in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{S}), \quad \text{as } p \to +\infty.$$

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Positive solutions

Theorem [D.- Ianni - Pacella, Lond. Math. Soc. L.N. 2017], [D.- Grossi - Ianni - Pacella, 2018]

Let $(u_p)_p$ be a family of positive solutions of $(*)$ satisfying (F).
Then there exist a sequence $p_n \to +\infty$ such that one has:

- ▷ $x_{1,\infty}, \dots x_{k,\infty}$ are distinct $(N = k)$ simple, isolated concentration points;
- ▷ $-\nabla_x H(x_{i,\infty}, x_{i,\infty}) + \sum_{i \neq \ell} \nabla_x G(x_{i,\infty}, x_{\ell,\infty}) = 0$
- ▷ $u_{p_n}(x_{i,p_n}) \to \sqrt{e}$ for any i, (in particular $\|u_{p_n}\|_\infty \to \sqrt{e}$) as $n \to +\infty$
- ▷ $p_n \int_\Omega |\nabla u_{p_n}|^2 dx \to k \cdot 8\pi e$ as $n \to +\infty$
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Examples

- ▷ If Ω is not simply connected: there exist solutions with an arbitrary large number k of concentration points
  [Esposito - Musso - Pistoia, J. Diff. Equ. 2006]

A priori estimates

Theorem [Kamburov-Sirakov, 2018]

Let $p_0 > 1$. There exists a constant $C = C(p_0, \Omega)$ such that for all $p \geq p_0$ ANY solution u_p of $(*)$ satisfies:

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Corollary [Kamburov-Sirakov, 2018]

Let $p_0 > 1$. If Ω is star-shaped, then for any $p \geq p_0$ there exists $C = C(p_0, \Omega) > 0$ such that ANY solution u_p of $(*)$ satisfies

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namely condition (F).

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Remark

In starshaped domains (in particular in convex domains) the asymptotic analysis holds without any further assumption.

Positive solutions in convex domains

Conjecture [Gidas - Ni - Nirenberg, Comm. Math. Phys. 1979]

If Ω is convex, there is only one positive solution to $(*)$ for any $p > 1$.

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This uniqueness conjecture holds:

- in a ball (uniqueness can be easily established by rescaling in view of the uniqueness for the initial value problem of the associated ODE) [Gidas - Ni - Nirenberg, Comm. Math. Phys. 1979]
- in domains close to a ball [Zou, Annali SNS 1994]
- if $p \in (1, p_1)$, p_1 close to 1 [Lin, Manuscripta Math. 1994]
- (+ nondegeneracy) for least energy solutions in convex domains of $\mathbb{R}^2$ [Lin, Manuscripta Math. 1994]
- if Ω is convex w.r.t. two orthogonal directions (not necessarily convex) [Dancer, J. Differential Equations 1988], [Damascelli - Grossi - Pacella, Ann. IHP 1999]

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The conjecture holds for p large

Theorem [D - Grossi - Ianni - Pacella, 2018]

If Ω is convex, then there exists $p^\star = p^\star(\Omega) > 1$ s.t. for any $p \geq p^\star$ $(*)$ admits a unique positive solution.

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Main point of the proof

It is enough to show that:

there exists $p^\star = p^\star(\Omega) > 1$ such that for any $p \geq p^\star$ any solution u_p of $(*)$ satisfies

$$\mathrm{m}(u_p) = 1$$

because then the thesis follows directy from the uniqueness of the Morse index-1 solution

Theorem [Grossi - Takahashi, J. Funct. Anal. 2018]

In convex domains

$$-\nabla H(x_{i,\infty}, x_{i,\infty}) + \sum_{i \neq \ell} \nabla_x G(x_{i,\infty}, x_{\ell,\infty}) = 0 \qquad \text{for any } i = 1, \dots, k$$

is solvable only if $k = 1$ and $x_{1,\infty}$ is a critical point of the Robin function.

$$G(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} - H(x,y), \qquad R(x) = H(x,x)$$

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As a consequence in convex domains for a family u_p of solutions to $(*)$ ~~satisfying (F)~~

we have:

- $k = 1$,
- $\mathcal{S} = \{x_\infty\}$,
- x_∞ is a critical point of the Robin function.

Idea of the proof

* It is enough to prove that

there exists $p^\star = p^\star(\Omega) > 1$ such that for any $p \geq p^\star$ any solution u_p of $(*)$ satisfies

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Idea of the proof

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* Let us suppose by contradiction that there exists a family $(u_p)_p$, $p \to +\infty$, of solutions of $(*)$ such that $\mathrm{m}(u_p) \neq 1$.

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* Let us suppose by contradiction that there exists a family $(u_p)_p$, $p \to +\infty$, of solutions of $(*)$ such that $\mathrm{m}(u_p) \neq 1$.

* By the asymptotic analysis and the fact that $\Omega \subset \mathbb{R}^2$ is convex we have: there exist a point $x_\infty \in \Omega$ and a subsequence $p_n \to +\infty$ such that:
  - $x^+_{p_n} \to x_\infty$ is a critical point of the Robin function;
  - $\|u_{p_n}\|_\infty \to \sqrt{e}$ as $n \to +\infty$
  - $p_n \int_\Omega |\nabla u_{p_n}|^2 dx \to 8\pi e$ as $n \to +\infty$;
  - $\sqrt{p_n} u_{p_n} \to 0$ in $C^2_{loc}(\bar{\Omega} \setminus \{x_\infty\})$ as $n \to +\infty$;
  - suitable rescalings of u_{p_n} (about $x^+_{p_n}$) converge to $\mathcal{U}$ solution to (RL)
  - $\exists C > 0$ such that for all $x \in \Omega$ $|x - x^+_{p_n}|^2 p_n |u_{p_n}(x)|^{p_n - 1} \leq C$;
  - $\exists C > 0$ such that for all $x \in \Omega$ $|x - x^+_{p_n}| p_n |\nabla u_{p_n}(x)| \leq C$.

Idea of the proof

* Next we consider the linearized problem at u_{p_n}

$$\begin{cases} -\Delta v = \mu\, p_n u_{p_n}^{p_n-1} v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \\ \|v\|_\infty = 1 \end{cases}$$

μ_{i,p_n} eigenvalues (counted with multiplicity)

v_{i,p_n} eigenfunctions

$\mathrm{m}(u_{i,p_n}) = \#\{i \in \mathbb{N} \,:\, \mu_{i,p_n} < 1\}$

* It is immediate to see that:

$$\mu_{1,p_n} = \frac{1}{p_n} < 1 \quad (\text{with } v_{1,p_n} = u_{p_n});$$

* The core of the proof consists in showing that:

$$\mu_{2,p_n} = 1 + 24\pi\eta_1\varepsilon_{p_n}^2 + o(\varepsilon_{p_n}^2), \quad \text{as } n \to +\infty$$

where η_1 is the first eigenvalue of the Hessian of the Robin function at x_∞;

* Since $\Omega \subset \mathbb{R}^2$ is convex x_∞ is the unique critical point of the Robin function and in particular it is a nondegenerate minimum point [Caffarelli-Friedman, 1985], so $\eta_1 > 0$ and in turn $\mu_{2,p_n} > 1$ for any $n \geq n^*$.
Therefore

$$\mathrm{m}(u_{p_n}) = 1 \quad \text{for } n \geq n^*,$$

which gives the desired contradiction. □

Without any assumption on Ω

Theorem [D.- Grossi - Ianni - Pacella, 2018]

Let (u_{p_n}) be a sequence of *1-peak solutions* concentrating about a critical point x_∞ of the Robin function, then for $n \geq n^*$

$$1 \leq \mathrm{m}(u_{p_n}) \leq 2.$$

Moreover if x_∞ is nondegenerate, then u_{p_n} is nondegenerate and $\mathrm{m}(u_{p_n}) = 1 + \mathrm{m}(x_\infty)$.

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Some previous analogous results

- $N \geq 3$: [Bahri-Li-Rey, Calc. Var. PDE, 1995], [Grossi-Pacella, 2005]
- $N = 2$, Liouville equation: [Gladiali-Grossi, 2009]

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$(\mathcal{P}_2^k)$ $\quad w_{i,p}(x) := p \frac{u_p(x_{i,p} + \varepsilon_{i,p} x) - u_p(x_{i,p})}{u_p(x_{i,p})} \underset{p \to +\infty}{\longrightarrow} \mathcal{U}(x) \quad \text{in } C^1_{loc}(\mathbb{R}^2)$

$(\mathcal{P}_3^k)$ there exists $C > 0$ such that:

$$\min_{i=1\dots,k} |x - x_{i,p}|^2 \, p|u_p(x)|^{p-1} \leq C \quad \text{for all } x \in \Omega$$

Moreover, given any other family of points $x_{k+1,p}$ it is impossible to extract a new sequence such that $(\mathcal{P}_0^{k+1})$, $(\mathcal{P}_1^{k+1})$, $(\mathcal{P}_2^{k+1})$ and $(\mathcal{P}_3^{k+1})$ hold.
Last, defining the concentration set as

$$\mathcal{S} = \{ \lim_{p \to +\infty} x_{i,p} \,|\, i = 1, \dots, k \} = \{ x_{1,\infty}, \dots, x_{N,\infty} \} \subset \bar{\Omega}$$

$$\sqrt{p}\, u_p \to 0 \quad \text{in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{S}), \quad \text{as } p \to +\infty.$$

Least energy sign-changing solutions

Theorem [Grossi - Grumiau -Pacella, Ann. IHP 2012]

Let u_p be a family of least energy sign-changing solutions to $(\mathcal{E}_p)$.
Then:

- $\|u_p\|_{L^\infty} \to \sqrt{e}$
- $p \int_\Omega |\nabla u_p|^2 dx \to 16\pi e$

Under an extra assumption:

- $x_{1,p} = x_p^+$, $x_{2,p} = x_p^-$ and $x_{1,\infty} \neq x_{2,\infty}$.

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Examples

▷ Solutions of this kind have been constructed in
[Esposito - Musso - Pistoia, Proc. Lond. Math. Soc. 2007]

Bubble tower phenomenon in the radial case

Theorem [Grossi - Grumiau - Pacella, J. Math. Pures Appl. 2014]

Let $(u_p)_p$ be a family of least en. sign-changing radial solutions of $(\mathcal{E}_p)$ in $B_1(0) \subset \mathbb{R}^2$. Then

$$w_p^+(x) = p\frac{u_p(\varepsilon_p^+ x) - u_p(0)}{u_p(0)} \underset{p\to+\infty}{\longrightarrow} \mathcal{U}(x) \quad \text{in } C^1_{loc}(\mathbb{R}^2)$$

$$w_p^-(x) = p\frac{u_p(\varepsilon_p^- x) - u_p(x_p^-)}{u_p(x_p^-)} \underset{p\to+\infty}{\longrightarrow} \mathcal{V}_\ell(x) \quad \text{in } C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$$

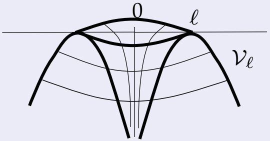
where

$$\begin{cases} -\Delta \mathcal{V}_\ell = e^{\mathcal{V}_\ell} - 4\pi\eta\delta_0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{\mathcal{V}_\ell} = 8\pi(1+\eta) \\ \mathcal{V}_\ell(x_\ell) = 0, \mathcal{V}_\ell \le 0 \\ \text{for a certain } \eta = \eta(\ell) \end{cases}$$

$$\mathcal{V}_\ell(x) := \log\left(\frac{2\alpha^2\beta^\alpha|x|^{\alpha-2}}{(\beta^\alpha+|x|^\alpha)^2}\right)$$

$$x_\ell = \lim_p x_p^-/\varepsilon_p^-, \quad \ell = |x_\ell|,$$

$$\alpha = \sqrt{2\ell^2+4}, \ \beta = \ell\left(\frac{\alpha+2}{\alpha-2}\right)^{1/\alpha}.$$



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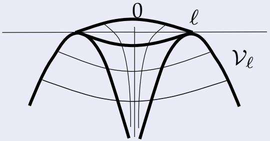

Moreover if r_p is the nodal radius of u_p, then

$$\frac{r_p}{\varepsilon_p^+} \to +\infty, \qquad \frac{r_p}{\varepsilon_p^-} \to 0 \qquad \text{as } p \to +\infty.$$

Bubble tower phenomenon in the symmetric case

Theorem [D - Ianni - Pacella, J. Eur. Math. Soc. 2015]

A bubble tower with two different bubbles appears also in G-symmetric domains by studying the asymptotic behavior of a special class of sign-changing solutions constructed in [D - Ianni - Pacella, J. Differ. Equ. 2013] and having two nodal regions and an interior nodal line.

Morse index of nodal radial solutions

Theorem [D - Ianni - Pacella, Math. Ann. 2017]

Let u_p be the least energy sign-changing radial solution of $(*)$ in $B_1(0) \subset \mathbb{R}^2$.
Then, for p sufficiently large, the Morse index:

$$m(u_p) = 12.$$

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Remark

$$m(u_p) = 12 = 1 + 11 = m(\mathcal{U}) + m(\mathcal{V}_\ell),$$

where $m(\mathcal{U})$ and $m(\mathcal{V}_\ell)$ are respectively the Morse indexes of the regular solution of (RL) and of the singular solution of (SL) [Chen - Lin, Comm. Pure Appl. Math 2010], to which $w_p^\pm$ converge.

Corollary: new (unexpected!) solutions

For p large there exist symmetric (but NOT radial) sign-changing solutions in the ball with interior nodal line.

- By a decomposition of the spectrum of the linearized operator

$$L_p = -\Delta - p|u_p|^{p-1}$$

  at the radial solution u_p, we prove that, for p large, among its negative eigenvalues there are at least:

  - 3 negative eigenvalues with G_4-symmetry
  - 3 negative eigenvalues with G_5-symmetry

  where by G_i, $i = 4, 5$, we mean the cyclic group of rotations by an angle of $\frac{2\pi}{i}$;
- the least energy nodal solution v_p^i in the space $H^1_{0,G_i}(B)$, $i = 4, 5$, has exactly two negative G_i-symmetric eigenvalues (following [Barsch - Weth, TMNA 2003]);
- then $v_p^i \neq u_p$, so v_p^i are NOT radial;
- by a result in [D - Ianni - Pacella, JDE 2013] we know that the nodal line of v_p^i does not touch the boundary, so they are not the least energy solutions [Aftalion - Pacella, C. R. Math. Acad. Sci. Paris 2004].

Bifurcation

Theorem [Gladiali - Ianni, 2017]

By analyzing the asymptotic behavior of u_p, as $p \to 1$, and its Morse index it is possible to prove that

$$m(u_p) = 6 \qquad \text{for } p \text{ close to } 1.$$

The change of Morse index from 6 to 12, as p increases, shows bifurcation.

Outline of the proof

- The number of negative eigenvalues $\mu_i(p)$ of the linearized operator

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the number of negative eigenvalues $\tilde{\mu}_i^n(p)$ of the weighted operator

$$\tilde{L}_p^n(p) = |x|^2(-\Delta - p|u_p|^{p-1}) \qquad \text{in } A_n.$$

- We decompose the spectrum of $\tilde{L}^n_p$ as:

$$\tilde{\mu}^n_i(p) = \tilde{\beta}^n_j(p) + \lambda_k, \qquad i, j = 1, 2, \ldots, \;\; k = 0, 1, 2, \ldots \tag{1}$$

where $\tilde{\beta}^n_j(p)$ are the eigenvalues of the 1-dimensional weighted operator

$$\tilde{L}^n_{p,rad} = r^2(-v'' - \frac{v'}{r} - p|u_p(r)|^{p-1}), \qquad r \in (\tfrac{1}{n}, 1)$$

and λ_k are the eigenvalues of the Laplace-Beltrami operator $-\Delta_{S^1}$ on the unit sphere

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- Take $n = n_p \to +\infty$ as $p \to +\infty$ and set $\tilde{\beta}_1(p) := \tilde{\beta}_1^{n_p}(p)$, then

$$\tilde{\beta}_1(p) \underset{p\to+\infty}{\longrightarrow} -\frac{\ell+2}{2} \simeq -26.9 \in (-36, -25)$$

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▷ The thesis follows from (1) and (2) (counting the eigenvalues with their multiplicity)

Crucial step: $\tilde{\beta}_1(p) \underset{p\to+\infty}{\longrightarrow} -\frac{\ell+2}{2}$

- ▷ $\tilde{\beta}_1(p) \underset{p\to+\infty}{\not\longrightarrow} -\infty$, then, up to a subsequence, $\tilde{\beta}_1(p) \to \tilde{\beta}_1$

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We want to pass to the limit

$$¿ \qquad \hat{\varphi}_p \longrightarrow \hat{\varphi} \geq 0; \ \not\equiv 0 \qquad ?$$

solution to

$$-\hat{\varphi}''(s) - \frac{\hat{\varphi}'(s)}{s} - e^{\mathcal{V}_\ell(s)}\hat{\varphi}(s) = \tilde{\beta}_1\frac{\hat{\varphi}(s)}{s^2}, \qquad s \in (0,+\infty) \tag{3}$$

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- From (3) we can compute $\tilde\beta_1$

Thank you for your attention!