Entire solutions for Liouville systems

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Entire solutions for Liouville systems

We look for solutions of the following **Liouville system**

$$\begin{cases} -\Delta u_1 = 2e^{u_1} + \mu e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = \mu e^{u_1} + 2e^{u_2} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_1} < +\infty \\ \int_{\mathbb{R}^2} e^{u_2} < +\infty \end{cases}, \qquad \text{(LS)}$$

with $\mu > -2$.

Entire solutions for Liouville systems

It is a generalization of the very well-known **Liouville equation**

$$\begin{cases} -\Delta u = e^u & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < +\infty \end{cases}. \tag{LE}$$

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Solutions of (LE) has been completely classified (Chen-Li '91):

$$u(x) = U_{\delta,y}(x) := \log \frac{64}{(8\delta + |x-y|^2)^2}.$$

Entire solutions for Liouville systems

If we look for **scalar solutions** of (LS), namely such that $u_1(x) \equiv u_2(x)$, they solve

$$\begin{cases} -\Delta u_i = (2+\mu)e^{u_i} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty. \end{cases};$$

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therefore we must have

$$u_1(x) = u_2(x) := U_{\mu,\delta,y}(x) = \log \frac{64}{(2+\mu)\left(8\delta + |x-y|^2\right)^2}.$$

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In the trivial case $\mu = 0$ the system is **decoupled**:

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Therefore, each component solves (LE) and we have a **6-parameter** family of solutions:

$$\begin{aligned} u_1(x) &= U_{\delta_1,y_1}(x) := \log \frac{32\delta_1}{(8\delta_1 + |x-y_1|^2)^2} \\ u_2(x) &= U_{\delta_2,y_2}(x) := \log \frac{32\delta_2}{(8\delta_2 + |x-y_2|^2)^2}. \end{aligned}$$

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The only exception being the case $\mu = -1$, corresponding to the **Toda system**:

$$\begin{cases} -\Delta u_1 = 2e^{u_1} - e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = -e^{u_1} + 2e^{u_2} & \text{in } \mathbb{R}^2 \end{cases}.$$

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Solutions have been completely classified (Jost-Wang '02).

Entire solutions for Liouville systems

The space of solutions is an **8-parameter** family:

$$u_1(x) = \log\left(64\delta\gamma\frac{64\delta^2 + 16\frac{\delta}{\gamma}|x-y_1|^2 + |x^2 - 2xy_2 + (y_1y_2 - y_3)|^4}{(64\delta^2 + 16\delta\gamma|x-y_2|^2 + |x^2 - 2xy_2 + (y_1y_2 + y_3)|^4)^2}\right)$$

$$u_2(x) = \log\left(64\frac{\delta}{\gamma}\frac{64\delta^2 + 16\delta\gamma|x-y_2|^2 + |x^2 - 2xy_2 + (y_1y_2 + y_3)|^4}{\left(64\delta^2 + 16\frac{\delta}{\gamma}|x-y_1|^2 + |x^2 - 2xy_2 + (y_1y_2 - y_3)|^4\right)^2}\right)$$

$$\delta, \gamma \in \mathbb{R}; \qquad y_1, y_2, y_3 \in \mathbb{C}.$$

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$$\delta,\gamma\in\mathbb{R}; \qquad y_1,y_2,y_3\in\mathbb{C}.$$

$$\gamma=1,\ y_1=y_2,\ y_3=0 \qquad \Rightarrow \qquad u_1\equiv u_2.$$

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Poliakovsky-Tarantello ('14, '16) gave sufficient conditions for existence of solutions of (LS) on the **masses**

$$\beta_i := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{u_i}.$$

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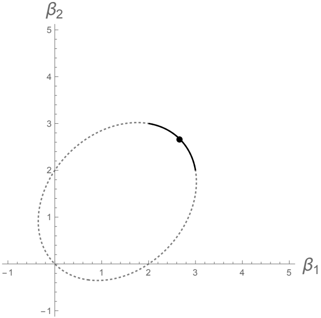
$$\beta_i := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i}.$$

Solutions exists if we assume

$$\begin{cases} \beta_1^2+\beta_2^2+\mu\beta_1\beta_2-2\beta_1-2\beta_2=0 \\ \beta_1+\frac{\mu}{2}\beta_2>1 \qquad \beta_2+\frac{\mu}{2}\beta_1>1 \\ (\beta_1-2)(\beta_2-2)\geq 0 \\ (\beta_1+\mu\beta_2)(\beta_2+\mu\beta_1)\geq 0 \end{cases}.$$

Entire solutions for Liouville systems



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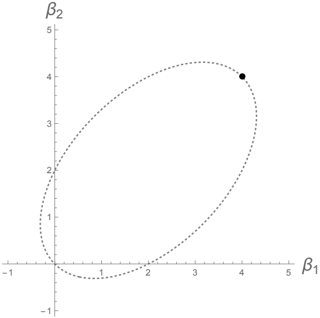

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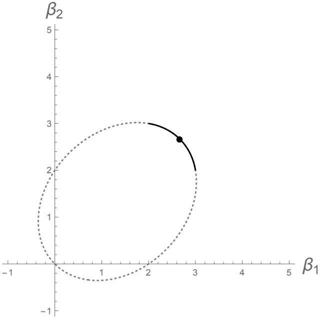

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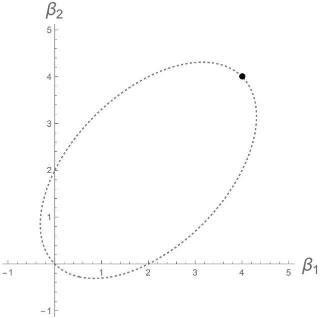

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Since scalar solutions verify $\beta_1 = \beta_2 = \dfrac{4}{2+\mu}$, they find in particular non-scalar solutions.

Bifurcation theory

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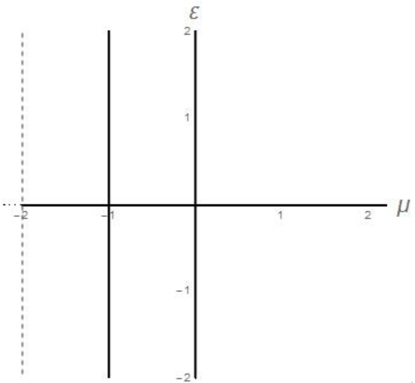

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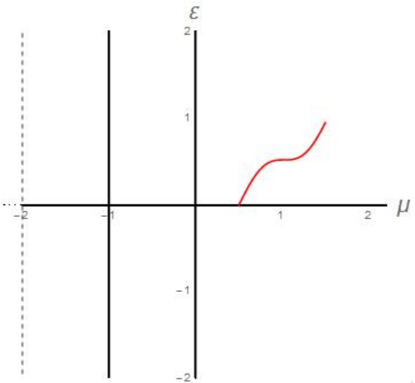

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By writing $(u_1, u_2) = \left(U_\mu + \frac{\Phi_1 + \Phi_2}{2}, U_\mu + \frac{\Phi_1 - \Phi_2}{2}\right)$, (Φ_1, Φ_2) must solve

$$\begin{cases} -\Delta\Phi_1 = (2+\mu)e^{U_\mu}\left(e^{\frac{\Phi_1+\Phi_2}{2}} + e^{\frac{\Phi_1-\Phi_2}{2}} - 2\right) & \text{in } \mathbb{R}^2 \\ -\Delta\Phi_2 = (2-\mu)e^{U_\mu}\left(e^{\frac{\Phi_1+\Phi_2}{2}} - e^{\frac{\Phi_1-\Phi_2}{2}}\right) & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{U_\mu} e^{\frac{\Phi_1+\Phi_2}{2}} < +\infty \\ \int_{\mathbb{R}^2} e^{U_\mu} e^{\frac{\Phi_1-\Phi_2}{2}} < +\infty \end{cases} . \tag{LS-Φ}$$

Bifurcation theory

We want to apply to (Φ_1, Φ_2) the following classical theorem:

Crandall-Rabinowitz, '71

If $T \in C^2((-2,2) \times X, Y)$ satisfies:

- $T(\mu, 0) = 0$ for all μ;
- $\ker(\partial_\Phi T(0,0)) = \mathrm{span}(w_0)$ and $\mathrm{R}(\partial_\Phi T(0,0))^\perp$ are both 1-dimensional;
- $\partial_{\mu,\Phi} T(0,0) w_0 \notin \mathrm{R}(\partial_\Phi T(0,0))$;

Then, there exists a non-trivial branch $(\mu(\varepsilon), \Phi^\varepsilon) : (-\varepsilon_0, \varepsilon_0) \to (-2,2) \times X$ such that $T(\mu(\varepsilon), \Phi^\varepsilon) = 0$ and $\left(\mu(0), \Phi^0\right) = (0,0)$.

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- Showing that $\ker(\partial_\Phi T(0,0))$ is 1-dimensional.

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Two main issues in applying Crandall-Rabinowitz theorem:

- Choosing suitable X, Y, T such that $T(\mu, \Phi) = 0$ implies Φ solves (LS-Φ);
- Showing that $\ker(\partial_\Phi T(0,0))$ is 1-dimensional.

To solve the first issue we may "move" the problem on the sphere using a stereographic projection.

The problem on the sphere

Consider the stereographic projection $\Pi : \mathbb{S}^2 \setminus \{0, 0, -1\} \to \mathbb{R}^2$:

$$\Pi : (x, y, z) \to \left(\sqrt{8}\frac{x}{1+z}, \sqrt{8}\frac{y}{1+z} \right).$$

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If (ϕ_1, ϕ_2) solves the (simpler) problem

$$\begin{cases} -\Delta_{\mathbb{S}^2}\phi_1 = 2\left(e^{\frac{\phi_1+\phi_2}{2}} + e^{\frac{\phi_1-\phi_2}{2}} - 2\right) \\ -\Delta_{\mathbb{S}^2}\phi_2 = 2\frac{2-\mu}{2+\mu}\left(e^{\frac{\phi_1+\phi_2}{2}} - e^{\frac{\phi_1-\phi_2}{2}}\right) \end{cases} \quad \text{on } \mathbb{S}^2, \qquad (\text{LS-}\mathbb{S}^2)$$

then $\left(\phi_1 \circ \Pi^{-1}, \phi_2 \circ \Pi^{-1}\right)$ solves (LS-Φ).

The problem on the sphere

Solutions of (LS-$\mathbb{S}^2$) are zeroes of the following smooth map $\mathcal{T}: W^{2,2}\left(\mathbb{S}^2\right) \times W^{2,2}\left(\mathbb{S}^2\right) \to L^2\left(\mathbb{S}^2\right) \times L^2\left(\mathbb{S}^2\right)$:

$$\mathcal{T}: (\mu, \phi_1, \phi_2) \to \begin{pmatrix} \Delta_{\mathbb{S}^2}\phi_1 + 2\left(e^{\frac{\phi_1+\phi_2}{2}} + e^{\frac{\phi_1-\phi_2}{2}} - 2\right) \\ \Delta_{\mathbb{S}^2}\phi_2 + 2\frac{2-\mu}{2+\mu}\left(e^{\frac{\phi_1+\phi_2}{2}} - e^{\frac{\phi_1-\phi_2}{2}}\right) \end{pmatrix}.$$

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Solutions of (LS-$\mathbb{S}^2$) are globally bounded, hence also solutions of (LS-Φ) are.

Therefore, since we have an L^∞-perturbation, the mass does not change:

$$\beta_1 = \beta_2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{U_\mu} = \frac{4}{2+\mu}.$$

The linearized operator

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The linearized operator in 0 has the form

$$\partial_\phi \mathcal{T}(\mu,0,0) : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \begin{pmatrix} \Delta_{\mathbb{S}^2} w_1 + 2w_1 \\ \Delta_{\mathbb{S}^2} w_2 + 2\dfrac{2-\mu}{2+\mu} w_2 \end{pmatrix};$$

therefore, the kernel is easy to compute, but it is too large.

The linearized operator

If $\mu \neq \mu_n := -2\frac{n^2+n-2}{n^2+n+2}$, then the kernel is **3-dimensional**:

$$\begin{pmatrix} w_1(\theta,z) \\ w_2(\theta,z) \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2}\cos\theta \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2}\sin\theta \\ 0 \end{pmatrix} \right.$$

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Anyway, these elements correspond to directions of scalar solutions, hence they do not satisfy the transversality condition and bifurcation always fails.

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If $\mu = \mu_n$, then the kernel is **2n + 4-dimensional**:

$$\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2}\cos\theta \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2}\sin\theta \\ 0 \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} 0 \\ P_n^0(z) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z)\cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z)\sin(m\theta) \end{pmatrix} \right\}_{m=1,\ldots,}$$

P_n^m are the **associated Legendre polynomials**:

$$P_n^m(z) = \frac{(-1)^m}{2^n n!} \left(1 - z^2\right)^{\frac{m}{2}} \frac{d^{n+m}}{dz^{n+m}} \left(z^2 - 1\right)^n .$$

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These new elements satisfy the transversality condition, but the kernel is too large!

The linearized operator

To get a 1-dimensional kernel we must restrict $\mathcal{T}$ to some suitable sub-spaces $\mathcal{X} \subset W^{2,2}\left(\mathbb{S}^2\right) \times W^{2,2}\left(\mathbb{S}^2\right)$, $\mathcal{Y} \subset L^2\left(\mathbb{S}^2\right) \times L^2\left(\mathbb{S}^2\right)$.

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We can restrict to the sub-spaces $\mathcal{X}_{\text{rad}}$, $\mathcal{Y}_{\text{rad}}$ of radial solutions.
In this case,

$$\ker(\partial_\phi \mathcal{T}(\mu, 0, 0)) = \text{span}\left\{ \begin{pmatrix} 0 \\ P_n^0(z) \end{pmatrix} \right\}$$

and we get a branch of radial solutions (Gladiali-Grossi-Wei '15).

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If we look for non-radial solutions we must exploit the symmetry properties of $\mathcal{T}$.

The role of symmetries

$-\Delta_{\mathbb{S}^2}$ is invariant under isometries, and in particular under the following:

$$\sigma : z \to -z, \qquad \rho_\alpha : \theta \to \theta + \alpha, \qquad \tau_\alpha : \theta \to -\theta + \alpha;$$

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The corresponding transformations of the plane (via the stereographic projection Π) are the following:

$$\widetilde{\sigma} : r \to \frac{8}{r}, \qquad \widetilde{\rho}_\alpha : \theta \to \theta + \alpha, \qquad \widetilde{\tau}_\alpha : \theta \to -\theta + \alpha.$$

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$\mathcal{T}$ is also odd with respect to the second component:

$$\begin{pmatrix} \mathcal{T}_1(\mu, \phi_1, -\phi_2) \\ \mathcal{T}_2(\mu, \phi_1, -\phi_2) \end{pmatrix} = \begin{pmatrix} \mathcal{T}_1(\mu, \phi_1, \phi_2) \\ -\mathcal{T}_2(\mu, \phi_1, \phi_2) \end{pmatrix}.$$

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Expoiting these symmetries, we want to find invariant subspaces for $\mathcal{T}$ such that

$$\ker(\partial_\phi \mathcal{T}(\mu_n, 0, 0)) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ P_n^n(z)\cos(n\theta) \end{pmatrix} \right\}.$$

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Let us consider the following spaces:

$$\begin{aligned} \mathcal{X}_n := \{(\phi_1, \phi_2) : \quad & \phi_1 \circ \sigma = \phi_1, \quad \phi_1 \circ \rho_{\frac{\pi}{n}} = \phi_1, \quad \phi_1 \circ \tau_{\frac{\pi}{n}} = \phi_1, \\ & \phi_2 \circ \sigma = \phi_2, \quad \phi_2 \circ \rho_{\frac{\pi}{n}} = -\phi_2, \quad \phi_2 \circ \tau_{\frac{\pi}{n}} = -\phi_2\} \end{aligned}$$

$$\begin{aligned} \mathcal{Y}_n := \{(\psi_1, \psi_2) : \quad & \psi_1 \circ \sigma = \psi_1, \quad \psi_1 \circ \rho_{\frac{\pi}{n}} = \psi_1, \quad \psi_1 \circ \tau_{\frac{\pi}{n}} = \psi_1, \\ & \psi_2 \circ \sigma = \psi_2, \quad \psi_2 \circ \rho_{\frac{\pi}{n}} = -\psi_2, \quad \psi_2 \circ \tau_{\frac{\pi}{n}} = -\psi_2\}. \end{aligned}$$

The role of symmetries

By the symmetry properties, $\mathcal{T} : \mathcal{X}_n \to \mathcal{Y}_n$ and

$$\ker(\partial_\phi \mathcal{T}(\mu_n, 0, 0)) \cap \mathcal{X}_n = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ P_n^n(z) \cos(n\theta) \end{pmatrix} \right\}.$$

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By the symmetry properties, $\mathcal{T} : \mathcal{X}_n \to \mathcal{Y}_n$ and

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A similar argument works for most generators of the kernel.
If $\mathbf{m} > \frac{\mathbf{n}}{\mathbf{3}}$, then we can find subspaces $\mathcal{X}_{n,m}$, $\mathcal{Y}_{n,m}$ such that $\mathcal{T} : \mathcal{X}_{n,m} \to \mathcal{Y}_{n,m}$ and

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The main result

So we get the following result:

B.-Gladiali-Grossi, 2017

For any $m, n \in \mathbb{N}$ with $\frac{n}{3} < m \leq n$ there exists a branch of solutions $(\mu(\varepsilon), u_1^\varepsilon, u_2^\varepsilon)$ of (LS) bifurcating from $(\mu_n, U_{\mu_n}, U_{\mu_n})$ and satisfying:

$$\begin{cases} u_1^\varepsilon(r,\theta) = U_{\mu_n}(r,\theta) + \varepsilon P_n^m\left(\frac{8-r^2}{8+r^2}\right)\cos(m\theta) + \varepsilon^2 Z_1^\varepsilon(r,\theta), \\ u_2^\varepsilon(r,\theta) = U_{\mu_n}(r,\theta) - \varepsilon P_n^m\left(\frac{8-r^2}{8+r^2}\right)\cos(m\theta) + \varepsilon^2 Z_2^\varepsilon(r,\theta), \\ \mu(0) = \mu_n, \qquad Z_1^\varepsilon, Z_2^\varepsilon \in L^\infty\left(\mathbb{R}^2\right). \end{cases}$$

The main result

Some remarks:

- The solutions verify

$$u_1^\varepsilon\left(\frac{8}{r},\theta\right)=\log\frac{r^4}{64}+\begin{cases} u_1^\varepsilon(r,\theta) & \text{if } n+m \text{ is even} \\ u_2^\varepsilon(r,\theta) & \text{if } n+m \text{ is odd}\end{cases},$$

$$u_1^\varepsilon\left(r,\theta+\frac{\pi}{m}\right)=u_2^\varepsilon(r,\theta), \qquad u_1^\varepsilon\left(r,-\theta+\frac{\pi}{m}\right)=u_2^\varepsilon(r,\theta).$$

In particular, $u_i^\varepsilon\left(r,\theta+\frac{2\pi}{m}\right)=u_i^\varepsilon(r,\theta)=u_i^\varepsilon(r,-\theta)$.

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In particular, $u_i^\varepsilon\left(r,\theta+\frac{2\pi}{m}\right) = u_i^\varepsilon(r,\theta) = u_i^\varepsilon(r,-\theta)$.

- Due to invariance under rotation of (LS), we can equivalently bifurcate in the direction $\begin{pmatrix} 0 \\ P_n^m(z)\cos(m\theta+\varphi) \end{pmatrix}$ for $\varphi \in \mathbb{S}^1$.

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- We get $n - \left\lfloor \frac{n}{3} \right\rfloor$ (non-equivalent) branches of non-radial solutions, plus a branch of radial solutions from Gladiali-Grossi-Wei.

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- We get $n - \left\lfloor \frac{n}{3} \right\rfloor$ (non-equivalent) branches of non-radial solutions, plus a branch of radial solutions from Gladiali-Grossi-Wei.
- If $\mu = \mu_1 = 0$ we have a (couple of) branch(es) of non-radial solutions and a radial branch, recovering locally the 6-parameter family of the decoupled system.
- If $\mu = \mu_2 = -1$ we have two (couples of) branches of non-radial solutions and a radial branch, recovering locally the 8-parameter family of the Toda system from Jost-Wang.

The derivatives of $\mu(\varepsilon)$

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As for $\mu''(0)$, the situation is not clear:

$$\mu''(0) = \underbrace{C_{m,n}}_{>0} \left(\int_0^1 (P_n^m(z))^4 \mathrm{d}z + 2 \int_{-1}^1 z (P_n^m(z))^2 \int_{-1}^z \frac{1}{y^2\left(1-y^2\right)} \int_y^1 x (P_n^m(x))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \right.$$

$$\left. + \int_{-1}^1 (z+2m) \left(\frac{1-z}{1+z}\right)^m (P_n^m(z))^2 \int_{-1}^z \frac{1}{(y+2m)^2\left(1-y^2\right)\left(\frac{1-y}{1+y}\right)^{2m}} \int_y^1 (x+2m)\left(\frac{1-x}{1+x}\right)^m (P_n^m(x))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \right)$$

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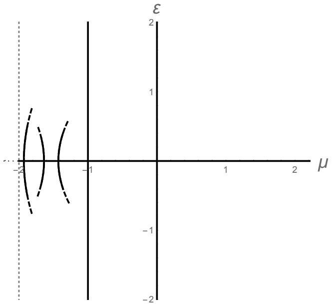
Scheme of the branches:



THANK YOU FOR YOUR ATTENTION!