Liouville Equations and Functional Determinants

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• While physicists may like these formulas, mathematicians usually have problems with infinite products of diverging numbers.

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If ζ is regular near $s = 0$ one can define the *regularized determinant* $\det'(-\Delta_g)$ via the following formula

$$\det{}'(-\Delta_g) = e^{-\zeta'(0)}.$$

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Existence of extremals is easy for positive genus. On spheres it can be achieved via a *balancing condition*, done in [Osgood-Phillips-Sarnak, '88] (see also [Aubin, '76], [Ghoussoub-Lin, '10], [Gui-Moradifam, '16]).

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Expanding the heat kernel (via parametrix) one can prove that

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} =: Tr(e^{\Delta t}) = \frac{1}{t} \sum_{j=0}^{l} t^j \int_{\Sigma} U_j(x) dV + O(t^l),$$

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therefore one gets bounds even in higher Sobolev norms.

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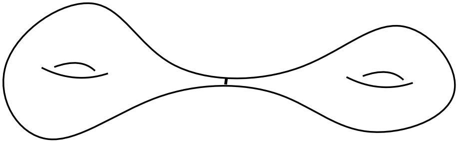

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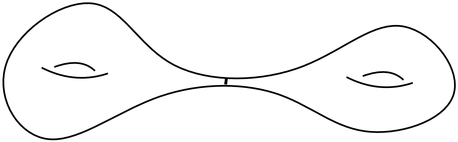

It was however shown in [Wolpert, '87] that

$$\det{}'(\hat{g}) \leq \frac{1}{l} e^{-\frac{c_1}{l}}; \qquad c_1 = c_1(\chi(\Sigma)),$$

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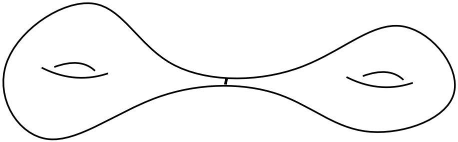

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Finally, a theorem in [Mumford, '71] shows that if l is bounded below and if $K_{\hat{g}} = const.$, then there is smooth convergence of the metrics.

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On S^2 it is known that the supremum of the k-th eigenvalue is $8\pi k$ ([Karpukhin-Nadirashvili-Penskoi-Polterovich '17]), with previous results in [Hersch , '70], [Petrides, '14], [Nadirashvili-Sire, '17] ($k = 1, 2, 3$).

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With few exceptions, no explicit formulas are known in higher genus.

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1. The *conformal Laplacian* for $n \geq 3$

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$$P_g \varphi = (-\Delta_g)^2 \varphi + \operatorname{div} \left[\left(\frac{2}{3} R g - 2 Ric \right) \circ \nabla \varphi \right], \qquad (a, b) = (0, 4)\,.$$

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• Both P_g and Q_g have a crucial role in the study of the topology of 4-manifolds (works by Chang, Gursky, Yang, Qing, . . .)

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• The theorem applies to L_g and $\mathcal{D}$, but not to the Paneitz operator P_g (discussed later).

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A consequence of these improved inequalities is that, for example, if $k_Q \in (8\pi^2, 16\pi^2)$ and if F_A is large, then the conformal volume must *concentrate* near a single point of M. One can then exploit the topology of M to find critical point of F_A of *saddle type*.

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Finally, using an integration by parts (Pohozaev), one shows that $\beta_i = 8\pi^2$ for all i, a contradiction to $k_Q \notin 8\pi^2\mathbb{N}$.

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On S^4 instead one has

$$\begin{aligned} F_P[w] &= \int_{S^4} \left[18(\Delta w)^2 + 64|\nabla w|^2 \Delta w + 32|\nabla w|^4 - 60|\nabla w|^2\right] dv \\ &+ 112\pi^2 \log\left(\fint_{S^4} e^{4(w-\overline{w})}\, dv\right). \end{aligned}$$

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- It goes similarly with compact hyperbolic manifolds.

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(d) A similar result holds in $\mathbb{R}^4$, much easier to prove.

A convenient change of variables

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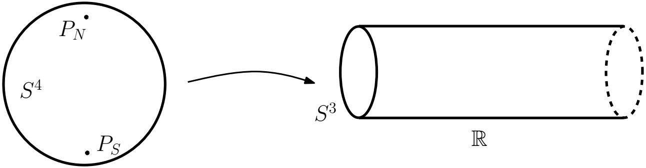
Once the north and south poles are removed, S^4 is conformally equivalent to the cylinder $S^3 \times \mathbb{R}$



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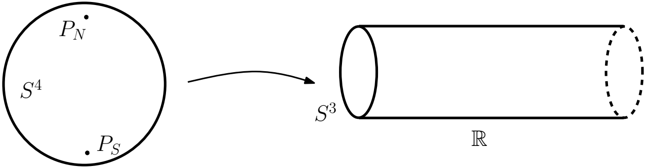

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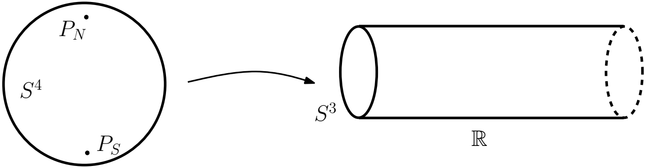

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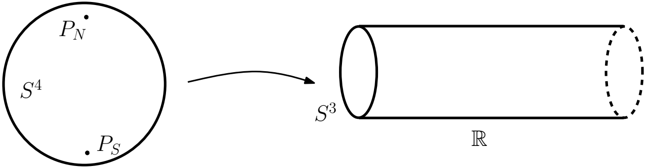

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- We can also assume that $u(t)$ is even in t.

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This is integrable, with a one-parameter family of periodic solutions.

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• The second formula reduces (E) to a third order, autonomous equation in u' (the exponential term disappears).

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<u>**Goal**</u> Look for solutions of (A) starting from the y-axis and converging asymptotically to the point $(1, 0, 0)$.

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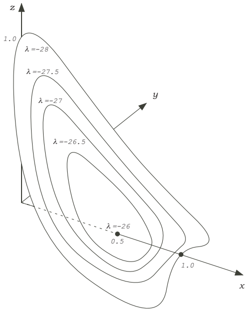

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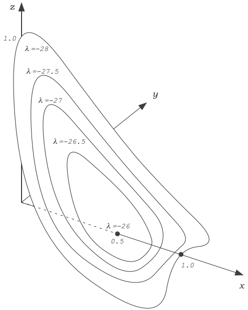

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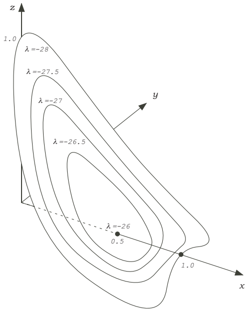

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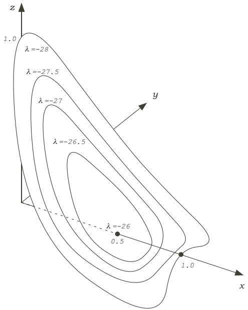

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For $\varepsilon > 0$, let $\overrightarrow{X}_\varepsilon(t)$ be the solution of (A) with initial data

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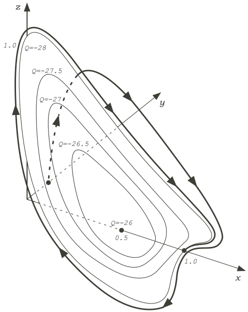

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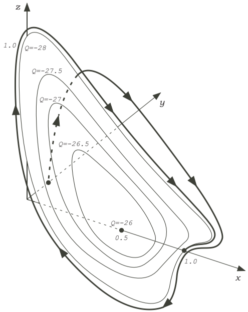

The proof uses refined asymptotic analysis, a Gronwall inequality and the construction of two (sort of) Lyapunov functions.

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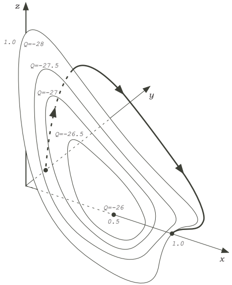

Technically, one needs to rule out infinitely-many oscillations.

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It is an interesting problem to find extremals of this quotient in $\mathbb{R}^4$.

The Euler equation

On $\mathbb{R}^4$ critical points satisfy

$$9\Delta^2 w + 32|\nabla^2 w|^2 - 32(\Delta w)^2 - 32\Delta u \; |\nabla u|^2 - 32\langle \nabla w, \nabla |\nabla w|^2 \rangle = 0.$$

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A natural question is whether a critical point always exists for F_P. This is be a natural counterpart of the Uniformization problems or the Yamabe problem. Apart from the compactness issues, new sharp Moser-Trudinger inequalities would be expected.

Thanks for your attention