Solutions and their total masses of Toda systems for general simple Lie algebras

Zhaohu Nie, Utah State Univ

Joint work with D. Karmakar, C.-S. Lin of NTU and J. Wei of UBC

Banff, April 3, 2018

Outline

Toda systems

I take it for granted that in this workshop, the general Toda systems are interesting. To each simple Lie algebra $\mathfrak{g}$ of rank n with Cartan matrix $(a_{ij})_{i,j=1}^n$, the associated Toda system on $\mathbb{R}^2$ with singular sources at the origin is

$$\begin{cases} \Delta u_i + 4\sum_{j=1}^{n} a_{ij}e^{u_j} = 4\pi\gamma_i\delta_0, & \gamma_i > -1, \\ \int_{\mathbb{R}^2} e^{u_i} < \infty, & 1 \le i \le n. \end{cases}$$

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Question: Can we classify all the solutions to the Toda system and what is the quantization result for the integrals, also called masses?

Liouville equation

The Liouville equation is the Toda system for the simplest simple Lie aglebra $A_1 = \mathfrak{sl}_2$ with Cartan matrix (2) and is

$$\begin{cases} \Delta u + 8e^u = 4\pi\gamma\delta_0, \quad \gamma > -1, \\ \int_{\mathbb{R}^2} e^u < \infty, \end{cases}$$

The solutions are classified by [Prajapat, Tarantello, 01] as

$$u(z) = 2\log \frac{|z|^\gamma}{\lambda + \frac{1}{\lambda}\left|\frac{z^{\gamma+1}}{\gamma+1} + c\right|^2}, \tag{1}$$

where $\lambda > 0$ and c is a complex number but is zero if γ is not an integer. The $\gamma = 0$ case was a result of [Chen, Li, 91].

There is also the quantization result $\displaystyle\int_{\mathbb{R}^2} e^u = \pi(1+\gamma)$.

Lie algebras and classifications of simple Lie algebras

Groups describe symmetry, and Lie groups describe continuous symmetries. Lie algebras are the linearizations of Lie groups whose Lie brackets reflect the non-commutativity of the multiplication of the corresponding Lie group.

Over $\mathbb{C}$, the simple Lie algebras (that is, Lie algebras with no nontrivial ideals) are classified into 4 infinite series and 5 exceptional types. Graphically they are presented by the Dynkin diagrams.

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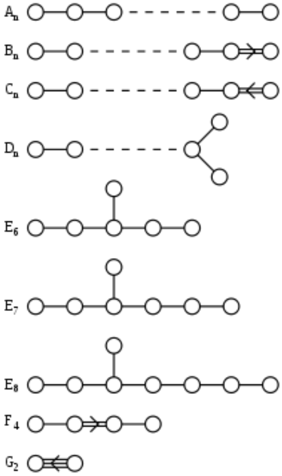

Cartan matrices

$$A_n = \mathfrak{sl}_{n+1} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \quad B_n = \mathfrak{so}_{2n+1} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix},$$

$$C_n = \mathfrak{sp}_{2n} : \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}, \quad D_n = \mathfrak{so}_{2n} : \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix},$$

$$G2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad F4 : \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix},$$

$$E6 : \begin{pmatrix} 2 & & -1 & & & \\ & 2 & & -1 & & \\ -1 & & 2 & -1 & & \\ & -1 & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \quad E7 : \begin{pmatrix} 2 & & -1 & & & & \\ & 2 & & -1 & & & \\ -1 & & 2 & -1 & & & \\ & -1 & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix},$$

and $E8$, where $a_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$.

Previous Results for A, B, C types

The solutions to the A_n Toda system are classified by [Jost, Wang, 02] without singular sources and by [Lin, Wei, Ye, 12] with singular sources at the origin.

The fundamental work [LWY] initiated the method of using an ODE involving W-invariants of Toda systems to classify solutions.

They have also established the non-degeneracy of the linearized system and the quantization result for the integrals

$$\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j} = \pi(2 + \gamma_i + \gamma_{n+1-i}), \quad 1 \leq i \leq n.$$

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[N., 16] generalized the classification to Toda systems of types B and C by treating them as reductions of type A.

General classification result in [KLNW]

For the Toda system associated to a general simple Lie algebra with finite masses and with singular sources at the origin,

1. The solution space is parametrized by a subgroup AN_Γ in a corresponding complex Lie group G. Here $N_\Gamma \subset N$ and A, N are the abelian and nilpotent subgroups in the Iwasawa decomposition $G = KAN$ (generalization of the Gram-Schmidt procedure).

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   When all the γ_i are integers, $N_\Gamma = N$, and the solution space has the maximal dimension, which is the same as the real dimension of the corresponding real Lie group.
   The solution space can be as small as A whose dimension is the rank n, and then all the solutions are radial with respect to the origin.

Formulas for the solutions

2. There are concrete expressions for the solutions

$$u_i = -\sum_{j=1}^{n} a_{ij} \log\langle j|\Phi^* C^* \Lambda^2 C\Phi|j\rangle + 2\gamma_i \log|z|, \quad 1 \leq i \leq n,$$

where $C \in N_\Gamma$ and $\Lambda \in A$, and $\Phi : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to N = N_- \subset G$ is the unique solution of

$$\begin{cases} \Phi^{-1}\Phi_z = \sum_{i=1}^{n} z^{\gamma_i} e_{-\alpha_i} \quad \text{on } \mathbb{C}\backslash\mathbb{R}_{\leq 0}, \\ \lim_{z\to 0} \Phi(z) = Id. \end{cases}$$

The $\langle j| \cdot |j\rangle$ is the highest matrix coefficient in the jth fundamental representation, and $*$ for classical Lie algebras is the conjugate transpose.

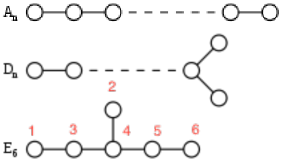
The total masses

3. The masses satisfy

$$\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j} = \pi(2 + \gamma_i - \kappa\gamma_i), \quad 1 \le i \le n,$$

where κ is the longest element in the Weyl group which maps all the positive roots to the negative roots. If $-\kappa\alpha_i = \alpha_k$, then $-\kappa\gamma_i := \gamma_k$. $-\kappa = Id$ except three cases where they are outer automorphisms of the Lie algebras represented by the symmetries of the Dynkin diagrams:

- A_n: $\alpha_i \leftrightarrow \alpha_{n+1-i}$;
- D_{2n+1}: $\alpha_{2n} \leftrightarrow \alpha_{2n+1}$;
- E_6: $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$.



C_2 Toda system as an example

For concreteness, just consider the C_2 Toda system.

$$\begin{cases} \Delta u_1 + 4(2e^{u_1} - e^{u_2}) = 4\pi\gamma_1\delta_0 \\ \Delta u_2 + 4(-2e^{u_1} + 2e^{u_2}) = 4\pi\gamma_2\delta_0 \\ \int_{\mathbb{R}^2} e^{u_1} < \infty, \ \int_{\mathbb{R}^2} e^{u_2} < \infty. \end{cases}$$

We use $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, and $\Delta = 4\partial_z\partial_{\bar z}$ to write the system as

$$\begin{cases} U_{1,z\bar z} + e^{u_1} = \pi\gamma^1\delta_0 \\ U_{2,z\bar z} + e^{u_2} = \pi\gamma^2\delta_0 \\ \int_{\mathbb{R}^2} e^{u_1} < \infty, \ \int_{\mathbb{R}^2} e^{u_2} < \infty, \end{cases}$$

where $\gamma^i = \sum_j a^{ij}\gamma_j$ and (a^{ij}) is the inverse Cartan matrix.

The Φ for C_2 Toda

The Φ is the most important information to obtain the solution.
Here $G = Sp_4\mathbb{C}$, and the solution to

$$\begin{cases} \Phi^{-1}\Phi_z = \sum_{i=1}^{2} z^{\gamma_i} e_{-\alpha_i} = \begin{pmatrix} 0 & & & \\ z^{\gamma_1} & 0 & & \\ & z^{\gamma_2} & 0 & \\ & & -z^{\gamma_1} & 0 \end{pmatrix} \\ \Phi(0) = Id \end{cases}$$

is

$$\Phi(z) = \begin{pmatrix} 1 & & & \\ \frac{z^{\mu_1}}{\mu_1} & 1 & & \\ \frac{z^{\mu_1+\mu_2}}{\mu_2(\mu_1+\mu_2)} & \frac{z^{\mu_2}}{\mu_2} & 1 & \\ -\frac{z^{2\mu_1+\mu_2}}{\mu_1(\mu_1+\mu_2)(2\mu_1+\mu_2)} & -\frac{z^{\mu_1+\mu_2}}{\mu_1(\mu_1+\mu_2)} & -\frac{z^{\mu_1}}{\mu_1} & 1 \end{pmatrix},$$

where $\mu_i = \gamma_i + 1 > 0$.

The Λ, C for C_2 Toda

The matrices $\Lambda \in A$ and $C \in N$ are

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_i > 0,$$

$$C = \begin{pmatrix} 1 & & & \\ c_{10} & 1 & & \\ c_{20} & c_{21} & 1 & \\ c_{30} & c_{31} & c_{32} & 1 \end{pmatrix}.$$

The blue ones are the coordinates, and c_{31} and c_{32} can be solved in them since C is symplectic.
Furthermore some c's are zero by considering the monodromy group decided by the γ_i, and this is our definition of N_Γ. For example, if $\gamma_1 = 0.5$ and $\gamma_2 = 1$, then the roots α_1 and $\alpha_1 + \alpha_2$ are not integers when the α_i are replaced by the γ_i, and hence c_{10} and c_{20} are zero. The other two roots α_2 and $2\alpha_1 + \alpha_2$ are integers.

The solutions for C_2 Toda

There are n fundamental representations for a simple Lie algebra of rank n whose highest weights are the n fundamental weights ω_i satisfying $\frac{2(\omega_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$. For classical Lie algebras, the ith fundamental representation is in the ith exterior product of the standard representations, except some spin representations for B_n and D_n. Therefore the highest matrix coefficients $\langle i| \cdot |i\rangle$ are just the leading principal minors of rank i of the matrices in the standard representations.

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With $\mathrm{X} = \Lambda C \Phi$, we see that

$$e^{-U_1} = |z|^{-2\gamma^1} (\mathrm{X}^* \mathrm{X})_{1,1},$$
$$e^{-U_2} = |z|^{-2\gamma^2} (\mathrm{X}^* \mathrm{X})_{[1,2],[1,2]},$$

where U_2 involves the leading principal 2×2 minor.

W-invariants

The W-invariants (also called characteristic invariants) are essential tools in our approach to the classification. They are polynomials W in the U_i and their derivatives with respect to z such that $W_{\bar{z}} = 0$ if the U_i are solutions. For example for the Liouville equation $U_{z\bar{z}} + e^{2U} = 0$, $W = U_{zz} - U_z^2$ is a W-invariant of homogeneous degree 2.

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[Feigin, Frenkel, 96] proved that there are n basic W-invariants for the Toda system associated to a simple Lie algebra of rank n. Furthermore the degrees d_j of the homogeneous basic invariants have Lie-theoretic meanings. [N., 14] gave a concrete construction of them for all simple Lie algebras and a direct proof that they are W-invariants.

The situation for Liouville equation

The classification in the genral case is to generalize the following strategies for the Liouville equation to any simple Lie algebra.

First, the solutions to the Liouville equation

$$\begin{cases} U_{z\bar{z}} + e^{2U} = \pi\frac{\gamma}{2}\delta_0, \quad \gamma > -1, \\ \int_{\mathbb{R}^2} e^{2U} < \infty, \end{cases}$$

locally on an open set D are

$$U(z) = \log\frac{|f'|}{1+|f|^2},$$

where f is a holomorphic function on D (called the developing map) and f' is nowhere zero.

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Secondly, the W-invariants for the local solutions are

$$W = U_{zz} - U_z^2 = \frac{1}{2}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \right),$$

one half of the Schwarzian derivative of the function f.

Restrict the W-invariants and the solutions

Thirdly, the Brezis-Merle estimate with the finite integral condition determine the simple form

$$W_j = \frac{w_j}{z^{d_j}},$$

using the Liouville theorem. The strength of the singularities shows that the right W-invariants are computed by choosing $f'(z) = z^{\gamma}$.

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These four steps of local solutions from holomorphic functions, relation with W-invariants, simple forms of W-invariants, and finally allowing multiplication by an element in the Lie group can all be done in the most general cases in Lie-theoretic ways.

Formula for Φ inspired by Kostant

The total mass calculation relies on the highest degree of z in the solution as illustrated in the Liouville equation (1). The degree information is contained in Φ. By establishing the close relationship of our work with [Kostant, 79] on Toda ODEs, we can write out the Φ in the enveloping algebra $U(\mathfrak{n}_-)$.

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Introduce the element $w_0 \in \mathfrak{h}$ in the Cartan subalgebra such that $\langle \alpha_i, w_0 \rangle = \mu_i$. Define $\mathcal{S}$ to be the set of all finite sequences

$$s = (i_1, i_2, \cdots, i_k), \quad 1 \le i_j \le n, \ k \ge 1,$$

$$e_{-s} = e_{-i_k} \cdots e_{-i_2} e_{-i_1} \in U(\mathfrak{n}_-),$$

$$\varphi(s, w_0) = \left\langle \sum \alpha_{i_j}, w_0 \right\rangle = \sum \mu_{i_j} > 0,$$

$$p(s, w_0) = (\mu_{i_1} + \cdots + \mu_{i_k}) \cdots (\mu_{i_{k-1}} + \mu_{i_k}) \mu_{i_k}.$$

The result is $\Phi(z) = \sum_{s \in \mathcal{S}} \frac{z^{\varphi(s, w_0)}}{p(s, w_0)} e_{-s}, \quad$ for $z \in \mathbb{C} \backslash \mathbb{R}_{\le 0}$.

Asympototic expansion and quantization result

The Lie-theoretic reason is that if ω is the highest weight of an irreducible representation, then $\kappa\omega$ is the lowest, and in most cases this is just $-\omega$. By the formula for U_i, the highest degree for z in U_i comes from the e_{-s} to reach from the highest weight to the lowest, and as such

$$\varphi(s, w_0) = \langle \omega_i - \kappa\omega_i, w_0 \rangle.$$

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$$\varphi(s, w_0) = \langle \omega_i - \kappa\omega_i, w_0 \rangle.$$

Then the solutions U_i satisfy that

$$U_i(z) = 2(\gamma^i - \langle \omega_i - \kappa\omega_i, w_0 \rangle) \log |z| + O(1), \quad \text{as } z \to \infty,$$

Form here, we get the quantization result

$$\frac{1}{\pi} \int_{\mathbb{R}^2} e^{u_i} = \langle \omega_i - \kappa\omega_i, w_0 \rangle.$$

Blowup masses

The local masses of the Toda system is defined by the local version on $B_1(0)$

$$\Delta u_i + 4\sum_{j=1}^{n} a_{ij} h_j e^{u_j} = 4\pi\gamma_i\delta_0.$$

For a sequence of solutions $u^k = (u_1^k, \cdots, u_n^k)$ blowing up at the origin, that is, $\lim_{k\to\infty} \max_{1\le i\le n} (u_i^k(z) - 2\gamma_i \log|z|)\big|_{z=0} = \infty$, the local mass is defined by

$$\sigma_i = \lim_{r\to 0}\lim_{k\to\infty} \frac{1}{\pi}\int_{B_r(0)} h_i e^{u_i^k}.$$

The set of local masses has important implications in the study of mean field equations of Toda type.

Blowup masses and the Weyl group

A general expectation is that the set of blowup masses corresponds to the Weyl group of the corresponding Lie algebra.

[Lin, Yang, Zhong] for A, B, C types, work in progress in general

The set of blowup masses is

$$\{(\langle\omega_i - s\omega_i, w_0\rangle, \cdots, \langle\omega_n - s\omega_n, w_0\rangle) \mid s \in W\},$$

where W is the Weyl group of the Lie algebra, generated by simple reflections in the simple roots

$$s_j(\beta) = \beta - \frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)}\alpha_j, \quad \forall \beta \in \mathfrak{h}^*.$$

The proof uses the total masses for the blowup profile which is an entire solution of a possibly smaller Toda system since some solutions may go to $-\infty$.

Examples demonstrating these masses

In a work in progress, I have found examples demonstrating all these blowup masses for all simple Lie algebras.
The construction uses the formula for the general solution by choosing suitable $\Lambda \in A$ corresponding to the $\lambda > 0$ in the Liouville case (1). The proof generalizes the degree considerations for the total masses.

Thanks for your attention!