Quantum Field Theory and Factorization Algebras

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1 Overview

Costello and Gwilliam [2] developed a general framework of constructing factorization algebra of quantum observables for any quantum field theory as defined earlier in Costello's book [1].

The Focused Research Group aimed to develop the formalism further toward two goals:

1) capturing bigger generality;

2) allowing concrete computations

The principal focus was to understand various sorts of defects of quantum field theory within the framework. In this report, we record the two main topics that were discussed in the meeting. These in particular include recent developments and open problems, presentation highlights, and scientific progress made.

2 Semiclassical OPE

2.1 OPE of Local Operators

This subsection is a quick summary of Section 10.3 of Costello and Gwilliam [2]. The main result describes the first order in $\hbar$ contribution to the operator product expansion (OPE) of local operators from the foundational construction of the quantum factorization algebra.

Consider classical field theory on $\mathbb{R}^n$ and classical observables $\mathrm{Obs}_0^{\mathrm{cl}}$ supported at $0 \in \mathbb{R}^n$. These are what we call local operators. For $\mathcal{O} \in \mathrm{Obs}_0^{\mathrm{cl}}$, we write $\mathcal{O}(x)$ for its translate to $x \in \mathbb{R}^n$. Then the claim is that the OPE, in the leading order in $\hbar$, gives a Poisson bracket on $\mathrm{Obs}_0^{\mathrm{cl}}$.

Let us understand this from the perspective of factorization algebras. Consider two local operators $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Obs}_0^{\mathrm{cl}}$. We look at a factorization product $\widetilde{\mathcal{O}}_1(0) \cdot \widetilde{\mathcal{O}}_2(x) \in \mathrm{Obs}^{\mathrm{q}} q(D(0, 2|x|))$ modulo $\hbar^2$ where $\widetilde{\mathcal{O}}_i$ is a lift to quantum observables defined mod $\hbar^2$. The result modulo $\hbar^2$ is independent of the choice of lifts. As we can, modulo $\hbar$, extend it to $x = 0$, the OPE measures an obstruction to extending this continuously across the origin $x = 0$.

Now we write

$$\widetilde{\mathcal{O}}_1(0) \cdot \widetilde{\mathcal{O}}_2(x) \equiv \hbar \sum_i \mathcal{O}^i(0) F_i(x) + (\text{regular at } x) \qquad \text{modulo } \hbar^2$$

where $\mathcal{O}^i(0)$ is a basis of operators and F_i is an analytic function modulo functions continuous at 0, which we denote by $F_i \in C^\omega(\mathbb{R}^n \setminus 0)/C^0(\mathbb{R}^n)$. Note that this information only depends on $\mathcal{O}, \mathcal{O}'$ and classical field theory; this is a semi-classical computation. Still, this amounts to doing certain simple Feynman diagram computation.

Let us use notation $\{\mathcal{O}_1(0), \mathcal{O}_2(x)\} = \lim_{\hbar\to 0} \hbar^{-1}\mathcal{O}_1(0)\cdot\mathcal{O}_2(x)$ for the OPE. In fact, one obtains a map

$$\{-,-\}\colon \mathrm{Obs}_0^{\mathrm{cl}} \otimes \mathrm{Obs}_0^{\mathrm{cl}} \to \mathrm{Obs}_0^{\mathrm{cl}} \otimes (C^\omega(\mathbb{R}^n \setminus 0)/C^0(\mathbb{R}^n)).$$

The notation is justified because this satisfies the Leibniz rule

$$\{\mathcal{O}_1(0)\mathcal{O}_2(0), \mathcal{O}_3(x)\} = \mathcal{O}_1(0)\{\mathcal{O}_2(0), \mathcal{O}_3(x)\} + \mathcal{O}_2(0)\{\mathcal{O}_1(0), \mathcal{O}_3(x)\}.$$

2.2 Formalism for OPE

We want to record the underlying formalism for the above. As a first step, we present the result as a concrete construction. The formalism for extended defects is work in progress. We eventually hope to understand to what extent we can analogously discuss the quantum OPE.

Consider a classical field theory with space of fields $\mathcal{E}$. We have an (IR regulated) propagator $P \in \mathcal{D}(M^2, \mathcal{E}^{\boxtimes 2})$ such that $P|_{M^2\setminus\Delta} \in C^\infty(M^2 \setminus \Delta, \mathcal{E}^{\boxtimes 2})$. The classical interaction terms are given by $I = \sum_{k\geq 3} I_k$ where $I_k \in \mathrm{Sym}^k_{C^\infty_M}(\mathcal{J}(\mathcal{E})^\vee) \otimes_{D_M} \mathrm{Dens}_M \subset \mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k})_{S_k}$. In this subsection, we work with $M = \mathbb{R}^n$.

As an example, consider scalar theory on M, where $\mathcal{E} = C^\infty(M)$. The propagator is

$$P(x,y) = \int_0^L K_t(x,y)dt \qquad \text{where} \quad K_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{t}}$$

A generic interaction term may be of the form $I_k(\phi) = \int_M \partial_{i_1}\phi^{j_1}\cdots\partial_{i_l}\phi^{j_l}$ with $\sum_{m=1}^{l} j_m = k$, where we abuse the notation to write a density as an integral.

To first order in $\hbar$, the only diagrams that will contribute to the OPE of two local operators will be irreducible trees connecting the two. Fix such a diagram, with interaction terms $I_{k_1}, ..., I_{k_p}$ and $p+1$ propogators, $(p-1)$ of them linking the p interaction vertices together in a line, and one on each end which will contract with the local operators of which we are computing the OPE.

Before contracting with the local operators at each end, we have $k = \sum_{i=1}^{p} k_i - 2p$ external legs of the interaction vertices, and two external legs of propagators, one at each end. Thus, the amplitude of this diagram is given by

$$\widetilde{\mathcal{A}} \quad \in \quad \mathcal{D}(M^2 \times M^k, \mathcal{E}^{\boxtimes 2} \boxtimes (\mathcal{E}^!)^{\boxtimes k}) \cong \mathcal{D}(M^2, \mathcal{E}^{\boxtimes 2})\hat{\otimes}\mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k})$$

Here is a proposition. The restriction of $\widetilde{\mathcal{A}}$ to the compliment of the diagonal $\Delta = M \hookrightarrow M^2$ satisfies

$$\mathcal{A} := \widetilde{\mathcal{A}}|_{(M^2\setminus\Delta)\times M^k} \quad \in \quad C^\infty(M^2\setminus\Delta, \mathcal{E}^{\boxtimes 2})\hat{\otimes}\mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k}) \quad \subset \quad \mathcal{D}(M^2\setminus\Delta, \mathcal{E}^{\boxtimes 2})\hat{\otimes}\mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k})$$

Proof. Follows from the fact that the propogators are smooth away from the diagonal themselves, and that the interactions are strictly supported on the small diagonal, so that contraction with them can not propogate the singularities away from the diagonals. □

Now let us fix two local operators, which for simplicity we assume to be linear in the fields, and thus given by $\mathcal{O}_i \in \mathcal{J}(\mathcal{E})^\vee \otimes \delta_{x_i}$ for $x_i \in M$ and $i = 1, 2$. Let us allow the positions of the local operators to vary, or equivalently view the underlying elements of $\mathcal{J}(\mathcal{E})^\vee$ as flat families of local operators. This is where we used the assumption $M = \mathbb{R}^n$; otherwise one has to be more careful. Then their contraction with $\mathcal{A}$ yields an element $\langle \mathcal{O}_1 \boxtimes \mathcal{O}_2, \mathcal{A}\rangle \in C^\infty(M^2 \setminus \Delta)\hat{\otimes}\mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k})$.

Now, fix a point $w_0 \in M$, choose coordinates x, y on M^2 in a disk $U \times U$ around (w_0, w_0) and let $z = \frac{x-y}{2}$ and $w = \frac{x+y}{2}$ on M^2. We will consider the limit $z \to 0$ with fixed $w = w_0$ which will be the common value $x = y = w_0$ in the limit. Then we have the following proposition.

Fix an expansion

$$\langle \mathcal{O}_1 \boxtimes \mathcal{O}_2, \mathcal{A}\rangle|_{w=w_0} = \sum_j f_j(z) \otimes \mathcal{O}_{(j)} \quad \in C^\infty(U \setminus \{w_0\})\hat{\otimes}\mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k})$$

such that the functions f_j are algebraically linearly independent. Then for each j such that $f_j(z)$ is singular at $z=0$, the corresponding observable $\mathcal{O}_{(j)}$ is supported at w_0, that is,

$$\mathcal{O}_{(j)} \in \mathrm{Sym}^k(\mathcal{J}(\mathcal{E})^\vee) \otimes \delta_{w_0} \subset \mathcal{D}(M^k, (\mathcal{E}^!)^{\boxtimes k})$$

Instead of providing an abstract proof, let us consider an example illustrating a general feature:

$$F(z) = \int_{x\in\mathbb{R}^n} \left(\int_{t_1=0}^{L} t_1^{-n/2} e^{-|x|^2/t_1}\right)\left(\int_{t_2=0}^{L} t_2^{-n/2} e^{-|x-z|^2/t_2}\right)\varphi(x)$$

Here are some observations one can immediately make:

- For $t_1, t_2 > 0$, the integral over x is clearly convergent from having the exponential.
- Similarly, for x away from 0 and z, the integral over t is convergent near 0 from the exponential.
- (Above proposition) Each of the integrals does yield a singular function, but it yields a well-defined integral kernel for $z \neq 0$, so that the total integral is still convergent if $z \neq 0$.

Now, for z near zero, this integral will typically diverge. We have $\int_{t_1=0}^{L} t_1^{-n/2} e^{-|x|^2/t_1} = |x|^{2-n}\tilde{\Gamma}(n,x)$ where $\tilde{\Gamma}(n,x)$ is regular (and non-vanishing) at $x=0$.

However, since the rate of blow up of this integrand near 0 is polynomial, for some $d \in \mathbb{N}$ sufficiently large, if we write our field $\varphi(x) = \varphi_{\leq d}(x) + \varphi_{>d}$ where $\varphi_{\leq d}(x)$ is the d^{th} order power series expansion of φ at x and $\varphi_{>d}$ vanishes to order d, then in the corresponding decomposition $F(z) = F_{\leq d}(z) + F_{>d}(z)$ we have that $F_{>d}$ is regular at 0.

Thus, we see that the singular contributions all come from $F_{\leq d}$, but this functional only depends on the power series expansion of φ at 0 to order d, and thus is a local functional.

3 Koszul Duality for Factorization Algebras

3.1 Mathematical Background

The fundamental theorem of deformation theory, for instance, as articulated by Lurie in DAG X, says that over a field k of characteristic 0, there is an equivalence between the ∞-category of differential graded Lie algebras and the one of formal moduli problems. Roughly speaking, the functor $\Psi\colon \mathrm{Lie} \to \mathrm{Moduli}$ is given by

$$\Psi(\mathfrak{g})(R) = \mathrm{MC}(\mathfrak{m}_R \otimes \mathfrak{g})$$

for a commutative differential (non-positively) graded algebra R over k. Here $\mathfrak{m}_R$ is fixed by an augmentation $R \to k$ and MC stands for the space of solutions to the Maurer–Cartan equation. To put it another way, one can find

$$\Psi(\mathfrak{g})(R) = \underline{\mathrm{Map}}_{\mathrm{CAlg}^{\mathrm{aug}}}(C^\bullet_{\mathrm{CE}}(\mathfrak{g}), R) = \underline{\mathrm{Map}}_{\mathrm{Lie}}(\mathfrak{D}(R), \mathfrak{g})$$

where $C^\bullet_{\mathrm{CE}}\colon \mathrm{Lie} \to (\mathrm{CAlg}^{\mathrm{aug}})^{\mathrm{op}}$ is the Chevalley–Eilenberg cochain functor and $\mathfrak{D}\colon (\mathrm{CAlg}^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}$ is the Koszul duality functor that is the right adjoint to $C^\bullet_{\mathrm{CE}}$.

In the same paper, Lurie also discusses a formal moduli problem for an associative algebra. Indeed, the statement is that if k is a field (of arbitrary characteristic), there is an equivalence between the ∞-category of augmented $\mathbb{E}_1$-algebras over k and the one of formal $\mathbb{E}_1$-moduli problems. This time, the situation is more symmetric and the functor $\Psi\colon \mathrm{Alg}^{\mathrm{aug}} \to \mathrm{Moduli}^{(1)}$ is given by

$$\Psi(B)(A) = \underline{\mathrm{Map}}_{\mathrm{Alg}^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(B), A) = \underline{\mathrm{Map}}_{\mathrm{Alg}^{\mathrm{aug}}}(\mathfrak{D}^{(1)}(A), B)$$

where $\mathfrak{D}^{(1)}\colon (\mathrm{Alg}^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Alg}^{\mathrm{aug}}$ is the Koszul dualty functor. Recall that for an $\mathbb{E}_1$-algebra (or a homotopy associative algebra) A with an augmentation $\epsilon\colon A \to k$, one can define the Koszul dual algebra $\mathfrak{D}^{(1)}(A)$ as

$A^! = \underline{\mathrm{Hom}}_A(k,k)$ where k is understood as a left A-module. We want to read the result as saying that the Koszul dual algebra of an associative algebra A corepresents the "Maurer–Cartan functor", namely,

$$\text{“}\mathrm{MC}(\mathfrak{m}_A \otimes B) \simeq \underline{\mathrm{Map}}_{\mathrm{Alg}^{\mathrm{aug}}}(A^!, B)\text{”}$$

where $\mathfrak{m}_A$ is the augmentation ideal. To put it another way, by applying the Yoneda lemma, we can recover the Koszul dual algebra $A^!$ from the "Maurer–Cartan functor".

3.2 QFT Interpretation

We would like to understand the QFT interpretation of this statement.

Here is a general set-up. Suppose we have a field theory on $\mathbb{R} \times X$ and hence a factorization algebra $\mathcal{F}$ on it. Suppose that the theory is topological along $\mathbb{R}$. Then for each point $x \in X$, the factorization algebra $\mathcal{F}_x := \mathcal{F}|_{\mathbb{R}\times x}$ on $\mathbb{R}$ is an $\mathbb{E}_1$-algebra which plays the role of A in the above discussion. For simplicity, we assume that for the projection $\pi\colon \mathbb{R} \times X \to \mathbb{R}$, the pushforward $\pi_*\mathcal{F}$ is trivial, namely, $\pi_*\mathcal{F} \simeq \mathbb{C}$ (or $\mathbb{C}[[\hbar]]$ in the quantum case but this will be omitted below).

Note that there is a map of factorization algebras $\mathcal{F}_x \to \pi_*\mathcal{F}$ on $\mathbb{R}$, because for a small neighborhood U of x we have $\mathbb{R} \times U \hookrightarrow \mathbb{R} \times X$ giving $\mathcal{F}(\mathbb{R} \times U) \to \mathcal{F}(\mathbb{R} \times X)$. Now from the assumption, the map $\mathcal{F}_x \to \pi_*\mathcal{F}$ can be thought of as an augmentation. Physically speaking, this is a choice of a vacuum.

As B is another $\mathbb{E}_1$-algebra, let us think of it as the algebra of observables of a certain topological quantum mechanics. It remains to understand the meaning of $\mathrm{MC}(\mathcal{F}_x^0 \otimes B)$, where $\mathcal{F}_x^0 \subset \mathcal{F}_x$ is the augmentation ideal. The claim is that

$\mathrm{MC}(\mathcal{F}_x^0 \otimes B)$ is the space of ways of coupling the two theories.

Proving this in some generality would involve some nontrivial research work, so we will be content with providing some informal explanation for its few different aspects.

First of all, let us note that it has a classical analogue. That is, if A, B are Poisson algebras, then using a Lie bracket from a Poisson structure, a Maurer–Cartan element corresponds to coupling at the classical level.

Here is an example. Consider 4d Chern–Simons theory on $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$. Consider the system of free fermions on $y = z = 0$. The space of fields is $A \in \Omega^\bullet(\mathbb{R}^2)\widehat{\otimes}\Omega^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{gl}_n$ and $\psi = (\psi_i, \psi^j) \in \Omega^\bullet(\mathbb{R}_x, \mathbb{C}^n \oplus (\mathbb{C}^n)^*)$. We know that free fermions lead to the Clifford algebra $B = \mathrm{Cl}(\mathbb{C}^n)$ as the algebra of observables. Hence local observables of the product system are $C^\bullet_{\mathrm{CE},\hbar}(\mathfrak{g}[[z]]) \otimes \mathrm{Cl}(\mathbb{C}^n)$. Here $C^\bullet_{\mathrm{CE},\hbar}(\mathfrak{g}[[z]])$ is an $\mathbb{E}_1$-algebra which is quantization of the Chevalley–Eilenberg complex $(C^\bullet_{\mathrm{CE}}(\mathfrak{g}[[z]], d_{\mathrm{CE}})$.

Let us take its classical limit; the Clifford algebra becomes $\mathrm{Sym}((\mathbb{C}^n \oplus (\mathbb{C}^n)^*))$ with the induced bracket from the pairing $\langle -, - \rangle$; hence we end up with a DG Lie algebra

$$(C^\bullet_{\mathrm{CE}}(\mathfrak{g}[[z]]) \otimes \mathrm{Sym}((\mathbb{C}^n \oplus (\mathbb{C}^n)^*)), d_{\mathrm{CE}}, \langle -, - \rangle).$$

Then its Maurer–Cartan element corresponds to a map of DG Lie algebras $\mathfrak{g}[[z]] \to \mathrm{Sym}((\mathbb{C}^n \oplus (\mathbb{C}^n)^*))$. For instance, $X \mapsto \Phi_X = (\psi \mapsto \langle \psi, X_0 \cdot \psi \rangle)$ is a map of DG Lie algebras, where $X = \sum X_n z^n \in \mathfrak{g}[[z]]$ and we abuse the notation to write $\psi \in \mathbb{C}^n \oplus (\mathbb{C}^n)^*$. Under our correspondence, this defines a coupling $\int_{\mathbb{R}} \psi A \psi$.

Now let us formulate this in a more conceptual way. Let us assume that $(\mathcal{F}_x \otimes B, d_{\mathcal{F}_x \otimes B})$ is quantum observables for topological quantum mechanics. One should imagine $d_{\mathcal{F}_x \otimes B} = Q + \hbar\Delta$, where Δ is the BV differential. In particular, it is an $\mathbb{E}_1$-algebra which can also be regarded as a DG Lie algebra. If it is not topological, one has to keep track of the information of Hamiltonian, but the story will essentially be the same.

From the assumption, we have the local constancy along $\mathbb{R}$, which gives a quasi-isomorphism

$$\mathcal{F}_x \otimes B \simeq \Omega^\bullet(\mathbb{R}) \otimes \mathcal{F}_x \otimes B$$

of DG Lie algebras. Then a Maurer–Cartan element α of $\mathcal{F}_x \otimes B$ corresponds to a Maurer–Cartan element O_α of the right-hand side, which we expand as $O_\alpha = O_\alpha^{(0)} + O_\alpha^{(1)}$ where $O_\alpha^{(i)}$ is of the form degree i. Namely, we have

$$d_{\mathcal{F}_x \otimes B}\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \qquad \longleftrightarrow \qquad \begin{cases} d_{\mathcal{F}_x \otimes B} O_\alpha^{(0)} + \frac{1}{2}[O_\alpha^{(0)}, O_\alpha^{(0)}] = 0 \\ d_{\mathrm{dR}} O_\alpha^{(0)} + d_{\mathcal{F}_x \otimes B} O_\alpha^{(1)} + [O_\alpha^{(0)}, O_\alpha^{(1)}] = 0 \end{cases}$$

The expected claim is that $S_\alpha = \int_{\mathbb{R}} O_\alpha^{(1)}$ is a solution to quantum master equation if and only if α satisfies the Maurer–Cartan equation. Again, proving it in some generality is work in progress.

Moreover, the assumption that $\pi_* \mathcal{F}_x$ is trivial yields $\pi_*(\mathcal{F}_x \otimes B) \simeq B$. From this one can observe that we obtain $\alpha \in \mathcal{F}_x^0 \otimes B$ if and only if when we compactify to $\mathbb{R}$ we get the trivial deformation of the topological quantum mechanics.

In sum, $\mathrm{MC}(\mathcal{F}_x^0 \otimes B)$ realizes a deformation of the Lagrangian as claimed and $\mathcal{F}_x^!$ is operators of universal QM system we couple at $\mathbb{R} \times x$. This gives all possible ways to couple QM system.

In fact, for 4d Chern–Simons theory, if we want to couple 4d CS with some quantum mechanics with QM operator B, then most general coupling looks like $\sum_n \int (\partial_z^n A)^a \rho_a[n]$, where $\rho_a[n] \in B$. It turns out that for the coupled system to be anomaly-free, $\rho_a[n]$ should satisfy some relations, which one can check to be the relations of the Yangian $Y(\mathfrak{g})$.

References

[1] K. Costello, Renormalization and effective field theory, *Mathematical Surveys and Monographs* **170** (2011), American Mathematical Society, Providence, RI/

[2] K. Costello and O. Gwilliam, Factorization algebras in quantum field theory. Vol. 2, to be published in 2019.