The Euler-Kronecker constants of number fields

Amir Akbary
University of Lethbridge

ANTD XI
May 2019

Euler-Kronecker Constant

Definition

Let

$$\zeta_K(s) = \frac{\alpha_K}{s-1} + c_0(K) + c_1(K)(s-1) + c_2(K)(s-1)^2 + \cdots .$$

Then

$$\gamma_K = \frac{c_0(K)}{\alpha_K}$$

is called the Euler-Kronecker constant of K.

Ihara's prime counting function

Definition

For $x > 1$, set

$$\Phi_K(x) = \frac{1}{x-1} \sum_{N(\mathfrak{p})^k \leq x} \left(\frac{x}{N(\mathfrak{p})^k} - 1 \right) \log N(\mathfrak{p}).$$

Note

This is analogous to the de la Valle Poussin function

$$\sum_{n<x} \frac{\Lambda(n)}{n} - \frac{1}{x} \sum_{n<x} \Lambda(n).$$

Characteristic features of $\Phi_K(x)$

$$\Phi_K(x) = \frac{1}{x-1} \sum_{N(\mathfrak{p})^k \le x} \left(\frac{x}{N(\mathfrak{p})^k} - 1 \right) \log N(\mathfrak{p})$$

- It is a continuous function of x.
- The oscillating term in the explicit formula for $\Phi_K(x)$ has the form
$$-\frac{1}{2(x-1)} \sum_{\rho} \frac{(x^{\rho}-1)(x^{1-\rho}-1)}{\rho(1-\rho)}.$$

Ihara's Theorem

Theorem (Ihara, 2006)

(i) Assume the Generalized Riemann Hypothesis (GRH) for $\zeta_K(s)$*. Then there exist positive constants* c_1, c_2 *such that*

$$-c_1 \log |d_K| < \gamma_K < c_2 \log\log |d_K|.$$

(ii) We have

$$\gamma_K = \lim_{x \to \infty} (\log x - \Phi_K(x) - 1).$$

Ihara's Observation

$$\gamma_K = \lim_{x\to\infty}(\log x - \Phi_K(x) - 1)$$

$$\Phi_K(x) = \frac{1}{x-1}\sum_{N(\mathfrak{p})^k\leq x}\left(\frac{x}{N(\mathfrak{p})^k} - 1\right)\log N(\mathfrak{p})$$

The function $\Phi_K(x)$ is an "arithmetic approximation" of $\log x$. If the field K has many prime $\mathfrak{p}$ with small norm, then $\Phi_K(x)$ increases faster than $\log x$, at least for a while. Thus, for such K, the value of γ_K can be "conspicuously negative".

Example 1: Cyclic extensions of degree p contained in $\mathbb{Q}(\zeta_{p^2})$

- For odd prime p, let K_p be the unique cyclic extension of degree p over $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta_{p^2})$.
- K_p is totally real with $d_{K_p} = p^{2p-2}$.
- ℓ splits completely in $K_p \iff \ell^{p-1} \equiv 1 \pmod{p^2}$
  $\iff p$ is a Wieferich prime in base ℓ.

Example 1: Cyclic extensions of degree p contained in $\mathbb{Q}(\zeta_{p^2})$

- $W(p) = \{\ell < p;\ \ell^{p-1} \equiv 1 \pmod{p^2}\}$.
- The list of non-empty $W(p)$ with $p < 100$ is

$$W(11) = \{3\},\ W(43) = \{19\},\ W(59) = \{53\},$$

$$W(71) = \{11\},\ W(79) = \{31\},\ W(97) = \{53\}.$$

- $2 \in W(1093)$ and $2 \in W(3511)$.

Example 1: Cyclic extensions of degree p contained in $\mathbb{Q}(\zeta_{p^2})$

Euler–Kronecker constants of global fields and primes with small norms 445

Table 1.

p	γ_p''	γ_p'	ε_p
3	1.76673	1.76741	0.00270354
5	1.6981	1.69927	0.0122214
7	1.84553	1.84723	0.032591
11	−1.43302	−1.43032	0.0577191
13	0.468641	0.472016	0.107757
17	3.5781	3.58283	0.210134
19	4.53435	4.53974	0.25948
23	4.47064	4.47731	0.346256
29	2.32308	2.33163	0.46998
31	4.61964	4.62896	0.540857
37	5.6061	5.6175	0.70755
41	4.2761	4.28883	0.805977
43	−0.929757	−0.916538	0.81594
47	−2.6783	−2.66375	0.91587
53	6.05396	6.071	1.17309
59	0.428977	0.447956	1.30809
61	4.62301	4.64288	1.40864
67	6.03706	6.05918	1.6139
71	−12.8724	−12.8496	1.57591
73	5.99832	6.02267	1.81104
79	−3.85765	−3.83146	1.92486
83	1.21387	1.24177	2.10718
89	7.51911	7.54953	2.37227
97	−5.02725	−4.99428	2.54395
101	2.75934	2.79415	2.75782
103	−2.22423	−2.18885	2.7859
107	5.75378	5.79103	3.00361
109	5.59505	5.63306	3.07587
1069	−4.10435	−3.63507	51.7394
1087	−5.5176	−5.03975	52.7617
1091	−3.11201	−2.63214	53.0135
1093	−748.191	−747.74	46.4644
1097	3.54759	4.03061	53.4188
1103	7.84455	8.33062	53.8033
1109	−0.666736	−0.178118	54.0736
3499	9.81761	11.521	206.78
3511	−2423.07	−2421.45	185.836
3517	7.66195	9.37476	207.986

A Problem

Problem

Does $\gamma_{K(p)}$ possess an asymptotic distribution function?
Is it possible to construct a certain density function M, such that

$$\lim_{Y\to\infty} \frac{\#\{p \leq Y;\ \gamma_{K(p)} \leq z\}}{\#\{p \leq Y\}} = \int_{-\infty}^{z} M(t)dt?$$

Example 2: Quadratic fields

The following are due to Ihara under the assumption of GRH.

- For imaginary quadratic fields, $0 < \gamma_K < 1$ holds for $|d_K| \leq 43$, but $\gamma_K < 0$ for $d_K = -47, \ -56$. For example

$$-0.072 < \gamma_{\mathbb{Q}(\sqrt{-47})} < -0.053.$$

- For real quadratic fields, $0 < \gamma_K < 2$ holds for $d_K \leq 100$, but

$$-0.181 < \gamma_{\mathbb{Q}(\sqrt{481})} < -0.167.$$

Example 2: Quadratic fields

Theorem (Mourtada-Murty, 2015)

Assume GRH. Let $\mathcal{F}(Y)$ denote the set of the fundamental discriminants in the interval $[-Y, Y]$ and let $N(Y) = \#\mathcal{F}(Y)$. Then, there exists a probability density function M, such that

$$\lim_{Y\to\infty} \frac{1}{N(Y)} \#\{d \in \mathcal{F}(Y);\ \gamma_{\mathbb{Q}(\sqrt{d})} \leq z\} = \int_{-\infty}^{z-\gamma} M(t)dt.$$

Moreover, the characteristic function $\varphi_{F_\sigma}(y)$ of the asymptotic distribution function $F_\sigma(z) = \int_{-\infty}^{z} M(t)dt$ is given by

$$\varphi_{F_\sigma}(y) = \prod_p \left(\frac{1}{p+1} + \frac{p}{2(p+1)} \exp\left(-\frac{iy\log p}{p^\sigma - 1}\right) + \frac{p}{2(p+1)} \exp\left(\frac{iy\log p}{p^\sigma + 1}\right)\right).$$

Example 3: Cubic extensions of $\mathbb{Q}(\sqrt{-3})$

- $k = \mathbb{Q}(\sqrt{-3})$.
- $\mathfrak{O}_k = \mathbb{Z}[\zeta_3]$, $\zeta_3 = e^{\frac{2\pi i}{3}}$.
- Consider the set

$$\mathcal{C} := \{c \in \mathfrak{O}_k;\ c \neq 1 \text{ is square free and } c \equiv 1 \pmod{\langle 9 \rangle}\}.$$

- For $c \in \mathcal{C}$ consider the extension $k(c^{1/3})/k$.

Example 3: Cubic extensions of $\mathbb{Q}(\sqrt{-3})$

Theorem (A. - Hamieh, 2018)

Let $\mathcal{N}(Y)$ be the the number of elements $c \in \mathcal{C}$ with norm not exceeding Y. There exists a smooth function $M_1(t)$ such that

$$\lim_{Y\to\infty} \frac{1}{\mathcal{N}(Y)} \# \{c \in \mathcal{C} : \mathrm{N}(c) \le Y \text{ and } \gamma_{K_c} \le z\} = \int_{-\infty}^{\overline{z}} M_1(t)\, dt,$$

where $\overline{z} = z - \gamma_k$.

Example 4: Cyclotomic extensions $\mathbb{Q}(\zeta_q)$

- For prime q denote $\gamma_{\mathbb{Q}(\zeta_q)}$ by γ_q.
- $d_{\mathbb{Q}(\zeta_a)} = q^{q-2}$.
- Ihara's general bounds imply that, under GRH, there exist positive integers c_1 and c_2 such that

$$-c_1 q \log q < \gamma_q < c_2 \log q.$$

- Since primes of small norms in a cyclotomic field have size q so for q large we expect that $\gamma_q > 0$. So the lower bound $-c_1 q \log q$ appears to be far from optimal.

Example 4: Cyclotomic extensions $\mathbb{Q}(\zeta_q)$

Ihara's Conjecture

1) $\gamma_q > 0$.
2) For fixed $\epsilon > 0$ and q sufficiently large we have

$$\frac{1}{2} - \epsilon \leq \frac{\gamma_q}{\log q} \leq \frac{3}{2} + \epsilon.$$

Example 4: Cyclotomic extensions $\mathbb{Q}(\zeta_q)$

Theorem (Ford-Luca-Moree, 2014)

1) We have $\gamma_{964477901} = -0.1823\ldots$.
2) Under the assumption of the Hardy-Littlewood conjecture, there are infinitely many prime q *for which* $\gamma_q < 0$. *Moreover,*

$$\liminf_{q\to\infty} \frac{\gamma_q}{\log q} = -\infty.$$

Example 4: Cyclotomic extensions $\mathbb{Q}(\zeta_q)$

Hardy-Littlewood Conjecture

Suppose $\mathcal{A}$ is an admissible set (i.e., there is no prime p such that $p \mid n\prod_{i=1}^{s}(a_i n+1)$ for every $n \geq 1$). Then the number of primes $n \leq x$ such that $n, a_1 n+1, \cdots, a_s n+1$ are all prime $\gg x/(\log x)^{s+1}$.

An appearance of γ_q in studying some inequalities equivalent to the GRH

Notation

- $\varphi(n)$ is Euler's function.
- γ is the Euler-Mascheroni constant.
- p_i denotes the i-th prime
- (N_k) denotes the sequence of primorials, where

$$N_k = \prod_{i=1}^{k} p_i$$

  is the k-th primorial.

Nicolas' Criterion for the Riemann hypothesis

Nicolas' Criterion (1983)

The Riemann hypothesis is true if and only if there are at most finitely $k \in \mathbb{N}$ for which

$$\frac{N_k}{\varphi(N_k)\log\log N_k} \leq e^{\gamma}.$$

Question

Question

Can we develop a similar theorem for the Generalized Riemann Hypothesis (i.e., all the non-trivial zeros of $\zeta_K(s)$ are located on the critical line $\Re(s) = 1/2$.)?

Joint Work With Forrest Francis (UNSW, Canberra)

GRH Criterion

Theorem (A. -Francis, 2018)

Let $q \leq 10$ or $q = 12, 14$. The GRH for the Dedekind zeta function of $\mathbb{Q}(\zeta_q)$ is true if and only if there are at most finitely $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log \overline{N}_k))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,1)}.$$

The Primorials in $S_{q,a}$

- $S_{q,a} = \{n \in \mathbb{N} \,;\, p \mid n \implies p \equiv a \pmod q\}$
- The k-th *primorial in* $S_{q,a}$

$$\overline{N}_k \stackrel{\text{def}}{=} N_{q,a}(k) = \prod_{i=1}^{k} \overline{p}_i,$$

where $\overline{p}_i$ is the i-th prime in the arithmetic progression $a \pmod q$.

Mertens' Theorem in AP

Theorem (Williams/ Languasco and Zaccagnini)

Let $x \geq 2$ *and* $q, a \in \mathbb{N}$ *be coprime. Then,*

$$\prod_{\substack{p \leq x \\ p \equiv a \pmod q}} \left(1 - \frac{1}{p}\right) \sim \frac{C(q,a)}{(\log x)^{\frac{1}{\varphi(q)}}},$$

as $x \to \infty$, *where*

$$C(q,a)^{\varphi(q)} = e^{-\gamma} \prod_{p} \left(1 - \frac{1}{p}\right)^{\alpha(p;q,a)}$$

and

$$\alpha(p;q,a) = \begin{cases} \varphi(q) - 1 & \text{if } p \equiv a \pmod q, \\ -1 & \text{otherwise.} \end{cases}$$

GRH Criterion

Theorem (A. - Francis, 2018)

Let $q \leq 10$ or $q = 12, 14$. The GRH for the Dedekind zeta function of $\mathbb{Q}(\zeta_q)$ is true if and only if there are at most finitely $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log \overline{N}_k))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,1)}.$$

GRH Criterion

- Under the assumption of GRH, the proof uses an explicit formula involving the zeros of Dirilchlet L-functions for an auxiliary function.
- The proof relies on computation of

$$\mathcal{F}_q := \sum_{\chi} \sum_{\rho \in \mathcal{Z}(\chi')} \frac{1}{\rho(1-\rho)}$$

for certain values of q, which are closely related to γ_q.
- We have

$$\mathcal{F}_q = \sum_{\substack{d|q \\ d \neq 1}} \varphi^*(d) \log \frac{d}{\pi} + 2\gamma_q - \varphi(q)(\gamma + \log 2) - \log \pi + 2.$$

GRH Criterion

Theorem (A. - Francis, 2018)

Assume GRH for the Dedekind zeta function of $\mathbb{Q}(\zeta_q)$. Then there are at most finitely $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log \overline{N}_k))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,1)}$$

is satisfied if and only if

$$\limsup_{x\to\infty} \sum_{\chi} \sum_{\rho \in \mathcal{Z}(\chi')} \frac{x^{i\Im(\rho)}}{\rho(\rho-1)} < 2\mathcal{R}_{q,1}.$$

GRH Criterion

Notation

- $\mathcal{Z}(\chi) = \{\rho \in \mathbb{C} \ ; \ L(\rho, \chi) = 0, \ \Re(\rho) \geq 0$ and $\rho \neq 0\}$.
- χ' denotes the primitive Dirichlet character which induces the Dirichlet character χ.
- $\mathcal{R}_{q,1} = \#\{b \in (\mathbb{Z}/q\mathbb{Z})^\times \mid b^2 \equiv 1 \pmod{q}\}$.

GRH Criterion

Theorem (A. - Francis, 2018)

Assume GRH for the Dedekind zeta function of $\mathbb{Q}(\zeta_q)$. *Then there are at most finitely* $k \in \mathbb{N}$ *for which*

$$\frac{\overline{N}_k}{\varphi(\overline{N}_k)(\log(\varphi(q)\log \overline{N}_k))^{\frac{1}{\varphi(q)}}} \leq \frac{1}{C(q,1)}$$

is satisfied if and only if

$$\limsup_{x\to\infty} \sum_{\chi} \sum_{\rho \in \mathcal{Z}(\chi')} \frac{x^{i\Im(\rho)}}{\rho(\rho-1)} < 2\mathcal{R}_{q,1}.$$

GRH Criterion

- We speculate that

$$\limsup_{x\to\infty} \sum_{\chi} \sum_{\rho\in\mathcal{Z}(\chi')} \frac{x^{i\Im(\rho)}}{\rho(\rho-1)} = \sum_{\chi} \sum_{\rho\in\mathcal{Z}(\chi')} \frac{1}{\rho(1-\rho)} := \mathcal{F}_q.$$

- We have

$$\mathcal{F}_q = \sum_{\substack{d|q \\ d\neq 1}} \varphi^*(d) \log \frac{d}{\pi} + 2\gamma_q - \varphi(q)(\gamma + \log 2) - \log \pi + 2.$$

- Since γ_q does not get "conspicuously negative" we speculate that the number of q for which GRH is equivalent to a Nicolas type inequality is finite.