W-constraints in enumerative geometry

Nitin K. Chidambaram

University of Alberta

May 12, 2019

arXiv:1812.08738 joint work with G. Borot, V. Bouchard, T. Creutzig and D. Noschenko

Main question

Virasoro constraints and their cousins W-constraints appear in various enumerative contexts.
Can we find a general framework to understand and generate such constraints?

Virasoro constraints

Virasoro algebra : L_i, $i \in \mathbb{Z}$, $c \in \mathbb{C}$

$$[L_i, L_j] = (i - j)L_{i+j} + \frac{c}{12}(i^3 - i)\delta_{i+j,0}.$$

Virasoro constraints

Virasoro algebra : L_i, $i \in \mathbb{Z}$, $c \in \mathbb{C}$

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{c}{12}(i^3 - i)\delta_{i+j,0}.$$

Generating functions $Z(x_1, x_2, \cdots)$ of enumerative invariants often satisfy Virasoro constraints $L_i(x_a, \partial_{x_b})$:

$$L_i Z = 0,\ i \geq 0.$$

Examples of Virasoro constraints

Intersection numbers on $\overline{\mathcal{M}}_{g,n}$ $(2g+n-2>0)$

$$F_{g,n}(k_1,\cdots,k_n) := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}$$

Examples of Virasoro constraints

Intersection numbers on $\overline{\mathcal{M}}_{g,n}$ $(2g+n-2>0)$

$$F_{g,n}(k_1,\cdots,k_n) := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n}$$

$$Z = \exp\left(\sum_{\substack{g\geq 0, n\geq 1, 2g+n-2>0\\ k_i\geq 0}} \frac{\hbar^{g-1}}{n!} F_{g,n}(k_1,\cdots,k_n) x_{k_1}\cdots x_{k_l}\right)$$

is the unique solution to certain Virasoro constraints $L_i(x_a, \partial_{x_b})$.
Witten-Kontsevich, Dijkgraaf-Verlinde-Verlinde

Examples of Virasoro constraints

- (*Mironov-Morozov-Semenoff*) BGW tau function
- (*Faber-Shadrin-Zvonkine*) r-spin intersection theory
- (conj. *Eguchi-Hori-Xiong*) Gromov-Witten theory
- (*Milanov*) Eynard-Orantin topological recursion

Topological recursion

- Introduced by Eynard-Orantin ('07), generalized by Bouchard-Eynard('13)
- **Input** : spectral curve $(\mathcal{C}, x, \omega_{0,1}, \omega_{0,2})$
- $\mathcal{C}$: Riemann surface , x : meromorphic function on $\mathcal{C}$

Topological recursion

- Introduced by Eynard-Orantin ('07), generalized by Bouchard-Eynard('13)
- **Input** : spectral curve $(\mathcal{C}, x, \omega_{0,1}, \omega_{0,2})$
- $\mathcal{C}$: Riemann surface , x : meromorphic function on $\mathcal{C}$
- **Output** : Recursively constructs $\omega_{g,n}$ – meromorphic differentials on $\mathcal{C}^{\times n}$

Topological recursion

- Introduced by Eynard-Orantin ('07), generalized by Bouchard-Eynard('13)
- **Input** : spectral curve $(\mathcal{C}, x, \omega_{0,1}, \omega_{0,2})$
- $\mathcal{C}$: Riemann surface , x : meromorphic function on $\mathcal{C}$
- **Output** : Recursively constructs $\omega_{g,n}$ – meromorphic differentials on $\mathcal{C}^{\times n}$
- $\omega_{g,n}$ counts r-spin intersection numbers, Hurwitz numbers, Gromov-Witten invariants of toric threefolds etc. (all related to $\overline{\mathcal{M}}_{g,n}$)

Bouchard-Eynard topological recursion

Example (r-Airy curve)

$$\mathcal{C} = \mathbb{C}_z, \quad x = \frac{1}{r} z^r, \quad W_{0,1} = z^r dz \quad W_{0,2}(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}$$

The $\omega_{g,n}$ constructed by the topological recursion counts r-spin intersection numbers on $\overline{\mathcal{M}}_{g,n}^{(r)}$.

W-algebras

- W-algebras are "non-linear" extensions of Virasoro algebras (examples of vertex algebras)

W-algebras

- W-algebras are "non-linear" extensions of Virasoro algebras (examples of vertex algebras)
- **Example**: $W_3 = W(\mathfrak{sl}_3)$ is generated by L_i and W_j, $i, j \in \mathbb{Z}$

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{c}{12}(i^3 - i)\delta_{i+j,0}$$
$$[L_i, W_j] = (i-2j)W_{i+j}$$
$$[W_i, W_j] = a\delta_{i+j,0} + bL_{i+j} + d \sum_{m+n=i+j} L_m L_n$$

W-algebras

- W-algebras are "non-linear" extensions of Virasoro algebras (examples of vertex algebras)
- **Example**: $W_3 = W(\mathfrak{sl}_3)$ is generated by L_i and W_j, $i, j \in \mathbb{Z}$

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{c}{12}(i^3 - i)\delta_{i+j,0}$$
$$[L_i, W_j] = (i-2j)W_{i+j}$$
$$[W_i, W_j] = a\delta_{i+j,0} + bL_{i+j} + d \sum_{m+n=i+j} L_m L_n$$

Z for r-spin intersection numbers satisfy W-constraints.

Higher Airy Structures

Consider $\mathcal{D} = \mathbb{C}[\![\hbar, \hbar\partial_{x_i}, x_i]\!]_{i\in I}$ with Lie bracket $[\hbar\partial_{x_i}, x_j] = \hbar\delta_{i,j}$

grading : $\deg \hbar = 2,\ \deg \hbar\partial_{x_i} = 1 = \deg x_i$

Higher Airy Structures

Consider $\mathcal{D} = \mathbb{C}[\![\hbar, \hbar\partial_{x_i}, x_i]\!]_{i \in I}$ with Lie bracket $[\hbar\partial_{x_i}, x_j] = \hbar\delta_{i,j}$

grading : $\deg \hbar = 2$, $\deg \hbar\partial_{x_i} = 1 = \deg x_i$

Definition (Higher Airy Structures)

A higher Airy structure is a family of $\{H_i\}_{i \in I} \in \mathcal{D}$ such that

- $H_i = \hbar\partial_{x_i} - P_i$ where $\deg P_i \geq 2$,
- $\oplus_i \mathcal{D}.H_i$ is a Lie subalgebra of $\mathcal{D}$, i.e., $[H_i, H_j] = \sum_k g_{i,j}^k H_k$ where $g_{i,j}^k \in \mathcal{D}$.

Higher Airy Structures

Theorem (Kontsevich-Soibelman)

There exists a unique solution Z to the set of equations

$$H_i Z = 0, \forall i \in I,$$

of the form

$$Z = \exp\left(\sum_{\substack{g \geq 0, n \geq 1 \\ 2g+n-2>0 \\ k_i > 0}} \frac{\hbar^{g-1}}{n!} F_{g,n}(k_1, \cdots, k_n) x_{k_1} \cdots x_{k_l} \right)$$

called the partition function.

$F_{g,n}$ is constructed recursively on $(2g+n-2)$.

Examples

Theorem (BBCCN)

The partition function of the Bouchard-Eynard topological recursion coincides with the partition function of a higher Airy structure based on modes of $\oplus_i W(\mathfrak{gl}_{r_i})$.

In particular, this includes the case of r-spin intersection numbers a.k.a FJRW theory of the A_n singularity (*Adler-van Moerbeke*).

Examples

Theorem (BBCCN)

The partition function of the Bouchard-Eynard topological recursion coincides with the partition function of a higher Airy structure based on modes of $\oplus_i W(\mathfrak{gl}_{r_i})$.

In particular, this includes the case of r-spin intersection numbers a.k.a FJRW theory of the A_n singularity (*Adler-van Moerbeke*).

Corollary

The Bouchard-Eynard topological recursion is only well-defined (produces symmetric $\omega_{g,n}$) for admissible spectral curves.

Examples

- generating function of open intersection numbers (*Alexandrov*)

Examples

- generating function of open intersection numbers (*Alexandrov*)
- FJRW theory for simple singularities of type D and E (*Bakalov-Milanov*)

Remarks

We recover many of the existing W-constraints in the literature. In addition, we also construct many new ones.

- Enumerative meaning for other higher Airy structures of type *ADE*?

Remarks

We recover many of the existing W-constraints in the literature. In addition, we also construct many new ones.

- Enumerative meaning for other higher Airy structures of type ADE?
- r-spin open intersection numbers?

Thank you!