(De)synchronization for Markov-perturbation of synchronized random networks

Retreat for Young Researchers in Probabiltiy
and Areas of Applications

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Outline

- Markov Random Networks
- Deterministic random network: Synchronization
- Markov perturbations
- Large-probability Synchronization & Small-probability Desynchronization

Setting Up

k-state set: $\mathcal{S} = \{s_1, \cdots, s_k\}$

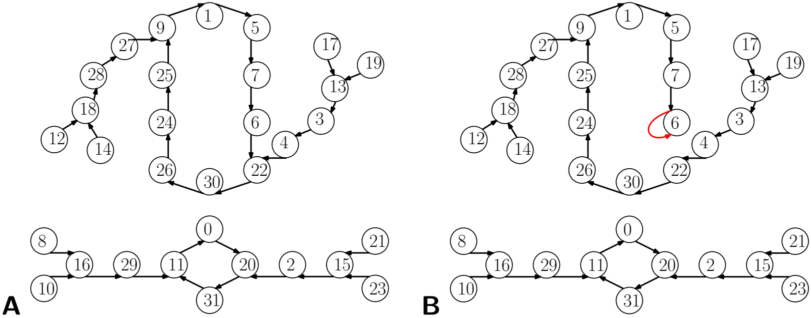
Physical networks on $\mathcal{S}$ are usually subjected to **noise influences**

Two types of noises: extrinsic noise, intrinsic noise

• **Extrinsic Noise:** environmentally or functionally related — macroscopic

Modeling extrinsic noise: **metric dynamical system** $(\Omega, \mathcal{F}, \mu, \theta)$
$(\Omega, \mathcal{F}, \mu)$: probability space
$\theta : \Omega \to \Omega$ invertible
μ ergodic θ-invariant measure

Extrinsic noise



A typical $\omega = (\cdots \overset{0}{B}\ \overset{1}{B}\ \overset{2}{A}\ \overset{3}{B}\ \overset{4}{B}\ \overset{5}{A}\ \overset{6}{A}\ \overset{7}{A}\ \overset{8}{B}\ \overset{9}{B}\ \overset{10}{A} \cdots) \in \Omega$

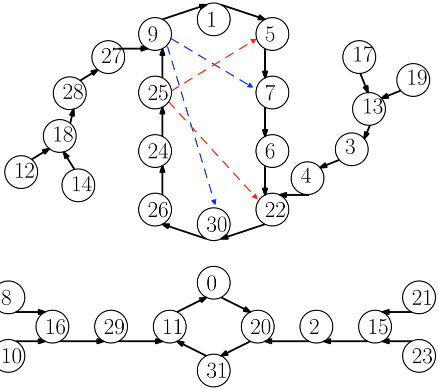
Random dynamics along the realization of ω :

$$\cdots \overset{10}{f_A} \circ \overset{9}{f_B} \circ \overset{8}{f_B} \circ \overset{7}{f_A} \circ \overset{6}{f_A} \circ \overset{5}{f_A} \circ \overset{4}{f_B} \circ \overset{3}{f_B} \circ \overset{2}{f_A} \circ \overset{1}{f_B} \circ \overset{0}{f_B} : \mathcal{S} \to \mathcal{S}$$

Intrinsic noise

Intrinsic Noise: internal uncertainties among individuals — microscopic
Modeling intrinsic noise: stochastic matrices

A + intrinsic noise:



Markov Random Network

Markov Random Network (MRN): A framework incorporating both extrinsic and intrinsic noises.

Mathematically: *a family $\mathcal{A}$ of stochastic processes* on a common probability space $(\Psi, \mathcal{G}, \mathbb{P})$ and take values in $\mathcal{S} \times \Omega$ s.t. for any process $X := \{X_n; n \in \mathbb{N}_0\} \in \mathcal{A}$:

- *(Stochasticity)* For μ-a.e. $\omega \in \Omega$, $n \geq m$,

$$\sum_{i=1}^{k} \mathbb{P}\Big\{X_n = (s_i, \theta^{n-m}\omega) | X_m = (s_j, \omega)\Big\} = 1, \quad \forall j \in \{1, 2, \cdots, k\}.$$

- *(Markov property)* For μ-a.e. $\omega \in \Omega$, $n \geq 0$,

$$\begin{aligned} & \mathbb{P}\big\{X_{n+1} = (s_{i_{n+1}}, \theta^{n+1}\omega) \,|\, X_n = (s_{i_n}, \theta^n\omega)\big\} \\ = \;& \mathbb{P}\big\{X_{n+1} = (s_{i_{n+1}}, \theta^{n+1}\omega) \,|\, X_0 = (s_{i_0}, \omega), \cdots, X_n = (s_{i_n}, \theta^n\omega)\big\}. \end{aligned}$$

- *(Common transitions)* For μ-a.e. $\omega \in \Omega$, the transition probabilities

$$p_{ij}(n, \omega) = \mathbb{P}\big\{X_n = (s_i, \theta^n\omega) | X_0 = (s_j, \omega)\big\}, \; i, j \in \{1, \cdots, k\}, n \in \mathbb{N}_0$$

  are in dependent of X, i.e. all processes from $\mathcal{A}$ share the same transition probabilities.

Markov cocycles

Transition probabilities (cocycle) of a MRN

$$\mathcal{P} := \{\mathcal{P}(n,\omega) = (p_{ij}(n,\omega))_{1\le i,j\le k} : \omega \in \Omega, n \in \mathbb{N}_0\}$$

is a **Markov cocycle** over Θ.

Cocycle property: for μ-a.e. $\omega \in \Omega$,

$$\mathcal{P}(n+m,\omega) = \mathcal{P}(m,\theta^n\omega)\cdot\mathcal{P}(n,\omega), \quad \forall n,m \in \mathbb{N}_0.$$

Any process $X := \{X_n; n \in \mathbb{N}_0\}$ from MRN is uniquely determined by its initial distribution:

$$\vec{p}(n,\omega) = \mathcal{P}(n,\omega)\vec{p}(0,\omega),$$

where $\vec{p}(n,\omega) = (p_i(n,\omega))_{i=1,\cdots,k}$ s.t.

$$p_i(n,\omega) = \mathbb{P}\{X_n = (s_i,\theta^n\omega) | X_0 = (\,\cdot\,,\omega)\}, \quad i \in \{1,\cdots,k\}.$$

Deterministic Random Networks

A special class of MRNs: **Deterministic Random Networks (DRNs)**

Deterministic transition cocycle :

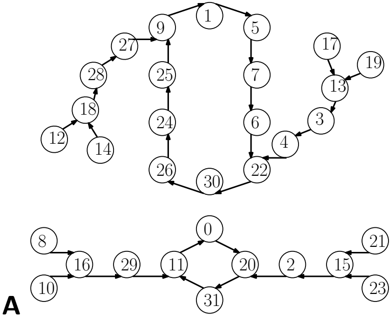
$$\text{Each } p_{ij}(n,\omega) \text{ is either 0 or 1}, \quad \mu\text{-a.e. } \omega \in \Omega.$$

DRN $\Longleftrightarrow$ dtds-RDS (*discrete-time, discrete-state random dynamical system*)

$\mathcal{P}$: transition cocycle of DRN, $\quad \mathcal{A}(n,\omega)$: dtds-RDS

$$\mathcal{A}(n,\omega)s_j = s_i \quad \text{iff} \quad p_{ij}(n,\omega) = 1, \quad i,j \in \{1,\cdots,k\}.$$

Deterministic Random Network



Synchronization

dtds-RDS $\mathcal{A}$ **synchronizes**: for μ-a.e. $\omega \in \Omega$, $\exists\ n(\omega) > 0$ s.t. $\forall i, j \in \{1, \cdots, k\}$,

$$\mathcal{A}(n, \omega)s_i = \mathcal{A}(n, \omega)s_j, \quad \forall n \geq n(\omega).$$

Contraction property: after finite times, all trajectories coincide

Neuron Network: reliability (response of the network remains the same independent of initial point).

Non-autonomous dynamical systems: random attractor

Markov Perturbations

$\mathcal{A}^0$: dtds-RDS (or a DRN), $\qquad$ $\mathcal{P}^0$: transition cocycle of $\mathcal{A}_0$

Markov-perturbation: A family $\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$ of MRNs is a perturbation of $\mathcal{A}^0$ if $\forall j \in \{1, \cdots, k\}$,

$$|\mathcal{P}^\varepsilon(1,\omega)\vec{e}_j - \mathcal{P}^0(1,\omega)\vec{e}_j| \le \varepsilon, \quad \mu\text{-}a.e.\ \omega \in \Omega,$$

where $\{\mathcal{P}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$ are transition cocycles of $\mathcal{P}^0$.

Questions: Assume that $\mathcal{A}^0$ is synchronized.

(1) Are $\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$ synchronized with large probability?

(2) If so, can one estimate the probability in terms of $\mathcal{P}^\varepsilon$?

(3) Can small-probability desynchronization really occur?

(4) If yes, under what conditions does desynchronization happen?

Convergence in distribution

$\mathcal{A}^0$: synchronized dtds-RDS
$\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$: Markov-perturbation of $\mathcal{A}^0$
$\{\mathcal{P}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$: transition cocycle of $\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$

Theorem A: $\forall \varepsilon \in [0, \varepsilon_0)$, $\exists$ invariant distribution $\vec{q}_\varepsilon(\cdot)$ of $\mathcal{A}^\varepsilon$, i.e., $\mathcal{P}^\varepsilon(n, \omega)\vec{q}_\varepsilon(\omega) = \vec{q}_\varepsilon(\theta^n \omega)$, μ-a.e. $\omega \in \Omega$, s.t.

- *(Convergence in distribution)* For $\forall$ initial distribution $\vec{p}(\cdot)$,

$$|\mathcal{P}^\varepsilon(n, \omega)\vec{p}(\omega) - \vec{q}_\varepsilon(\theta^n \omega)| \to 0, \quad n \to \infty.$$

- *(Continuity of invariant distribution)* $\exists\, i(\cdot) : \Omega \to \{1, \cdots, k\}$ s.t. for μ-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \vec{q}_\varepsilon(\omega) = \vec{e}_{i(\omega)} = \vec{q}_0(\omega).$$

Explicit expression of invariant distribution

$C^r(r \geq 1)$ Markov perturbation: $\{\mathcal{P}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$ entry-wise C^r w.r.t ε

Explicit formula for invariant distribution:

$$\vec{q}_\varepsilon(\omega) = \vec{e}_{i(\omega)} + \sum_{j=1}^{r} \frac{\varepsilon^j}{j!} \vec{x}^{(j)}(\omega) + \vec{y}^{(r)}(\omega; \varepsilon)\varepsilon^r, \quad \forall \varepsilon \in [0, \varepsilon_0),$$

where $\vec{x}^{(j)}(\,\cdot\,), j = 1, \cdots, r$, are explicitly computable depending on derivatives of $\mathcal{P}^\varepsilon$ and (backward-)synchronization time of $\mathcal{A}^0$

Large-probability synchronization

$\mathcal{A}^0$: a *uniform synchronized* dtds-RDS

$\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$: a *uniform* C^1 Markov-perturbation of $\mathcal{A}^0$.

Theorem B: $\exists\, C_0 > 0$ for which the following hold. $\forall \varepsilon \in [0, \varepsilon_0)$ and any two stochastic processes $X := \{X_n^\varepsilon; n \in \mathbb{N}_0\}, Y := \{Y_n^\varepsilon; n \in \mathbb{N}_0\}$ from $\mathcal{A}^\varepsilon$, there exists $N := N(\omega, \varepsilon; X, Y) > 0$ such that $\forall n \geq N$,

$$\mathbb{P}\Big\{X_n^\varepsilon = Y_n^\varepsilon | X_0^\varepsilon = (\,\cdot\,, \omega), Y_0^\varepsilon = (\,\cdot\,, \omega)\Big\} > 1 - C_0\varepsilon, \quad \mu\text{-}a.e.\ \omega \in \Omega.$$

Large-probability almost synchronization

$\mathcal{A}^0$: synchronized dtds-RDS

$\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$: C^1 Markov-perturbation of $\mathcal{A}^0$.

Theorem C: $\forall \eta > 0$, $\exists\ C_\eta > 0$, $\exists$ sets $E_{\omega,\eta} \subseteq \mathbb{N}$, μ-a.e. $\omega \in \Omega$, s.t.

- $\displaystyle \lim_{n\to\infty} \frac{\#|E_{\omega,\eta} \cap \{1, \cdots, n\}|}{n} > 1 - \eta,$
- $\forall \varepsilon \in [0, \varepsilon_0)$, for any two stochastic processes $X := \{X_n^\varepsilon; n \in \mathbb{N}_0\}, Y := \{Y_n^\varepsilon; n \in \mathbb{N}_0\}$ from $\mathcal{A}^\varepsilon$, $\exists N = N(\omega, \varepsilon; X, Y) > 0$ s.t.

$$\mathbb{P}\Big\{X_n^\varepsilon = Y_n^\varepsilon | X_0^\varepsilon = (\,\cdot\,, \omega), Y_0^\varepsilon = (\,\cdot\,, \omega)\Big\} > 1 - C_\eta \varepsilon, \quad \forall n \geq N, n \in E_{\omega,\eta}.$$

Small-probability desynchronization

$C^r (r \geq 2)$ Markov-perturbations,
$\mathcal{A}^0$: synchronized dtds-RDS
$\{\mathcal{A}^\varepsilon; \varepsilon \in [0, \varepsilon_0)\}$: $C^r (r \geq 2)$ Markov-perturbation of $\mathcal{A}_0$

$$\Omega_h := \{\omega \in \Omega : \vec{x}^{(\ell)}(\omega) = \vec{0}, \ell = 1 \cdots, h\}, \quad h = 1, \cdots, r$$

Theorem D: Assume $\exists\ 1 \leq h < r$ s.t. $0 < \beta^{(h)} := \mu(\Omega_h) < 1$. Then for $\forall \eta > 0$, $\exists\ C_\eta, c_\eta > 0, \varepsilon_\eta \in (0, \varepsilon_0)$, and disjoint subsets $E_{\omega,\eta}, F_{\omega,\eta} \subseteq \mathbb{N}$, μ-a.e. $\omega \in \Omega$, s.t.

- $\lim\limits_{n\to\infty} \dfrac{\#|E_{\omega,\eta} \cap \{1, \cdots, n\}|}{n} > \beta^{(h)} - \eta$,
- $\lim\limits_{n\to\infty} \dfrac{\#|F_{\omega,\eta} \cap \{1, \cdots, n\}|}{n} > 1 - \beta^{(h)} - \eta$,
- $\forall \varepsilon \in [0, \varepsilon_\eta)$, for any two stochastic processes $X := \{X_n^\varepsilon; n \in \mathbb{N}_0\}, Y := \{Y_n^\varepsilon; n \in \mathbb{N}_0\}$ from $\mathcal{A}^\varepsilon$, $\exists\ N = N(\omega, \varepsilon; X, Y) > 0$ s.t.

$$\mathbb{P}\Big\{X_n^\varepsilon = Y_n^\varepsilon | X_0^\varepsilon = (\cdot, \omega), Y_0^\varepsilon = (\cdot, \omega)\Big\} \geq 1 - C_\eta \varepsilon^h, \forall n \geq N, n \in E_{\omega,\eta}$$

$$\mathbb{P}\Big\{X_n^\varepsilon \neq Y_n^\varepsilon | X_0^\varepsilon = (\cdot, \omega), Y_0^\varepsilon = (\cdot, \omega)\Big\} \geq c_\eta \varepsilon^{h-1}, \forall n \geq N, n \in F_{\omega,\eta}$$

Conclusion

- Uniform synchronized DRN $+$ Uniform C^1 Markov perturbation

  $\Longrightarrow$ Large-probability synchronization

- Synchronized DRN $+$ C^1 Markov perturbation

  $\Longrightarrow$ Large-probability almost synchronization

- Synchronized DRN $+$ $C^r (r \geq 2)$ Markov perturbation

  $\Longrightarrow$ Large-probability almost synchronization $\&$ Small-probability almost desynchronization

Thank you for your attention