The relativistic semi-classical equation for a nucleon and its non-relativistic limit

Simona Rota Nodari

Institut de Mathématiques de Bourgogne, Université de Bourgogne

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Relativistic model: Coupled Dirac - Klein–Gordon equations

- one relativistic nucleon
- S: σ-meson $\rightsquigarrow$ medium range attractive interaction
- V: ω-meson $\rightsquigarrow$ short range repulsive interaction

Equation of the model:

$$\begin{cases} -i\boldsymbol{\alpha}\cdot\nabla\Psi + \beta(m+S)\Psi + V\Psi = (m-\mu)\Psi, \\ (-\Delta + m_\sigma^2)S = -g_\sigma^2 \Psi^* \beta \Psi, \\ (-\Delta + m_\omega^2)V = g_\omega^2 |\Psi|^2. \end{cases} \tag{DKG*}$$

- $\Psi : \mathbb{R}^3 \to \mathbb{C}^4$ quantum state of the particle
- $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\alpha}\cdot\nabla = \sum_{j=1}^{3} \alpha_j \partial_{x_j}$

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

- $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli matrices
- $0 < \mu < m$: energy of the particle

Non-relativistic limit

Equation of the model: $\Psi = \begin{pmatrix} \psi \\ \zeta \end{pmatrix}$

$$\begin{cases} -i\boldsymbol{\sigma}\cdot\nabla\zeta + (S+V+\mu)\psi = 0, \\ -i\boldsymbol{\sigma}\cdot\nabla\psi = (2m-\mu+S-V)\zeta, \\ (-\Delta + m_\sigma^2)S = -g_\sigma^2(|\psi|^2 - |\zeta|^2), \\ (-\Delta + m_\omega^2)V = g_\omega^2(|\psi|^2 + |\zeta|^2). \end{cases} \qquad \text{(DKG)}$$

Non-relativistic limit: $m, m_\sigma, m_\omega \to +\infty$ of the same order

Nuclear physics: $g_\sigma, g_\omega \to +\infty$ comparable to the masses

$$S \simeq -\frac{g_\sigma^2}{m_\sigma^2}(|\psi|^2 - |\zeta|^2) \quad V \simeq \frac{g_\omega^2}{m_\omega^2}(|\psi|^2 + |\zeta|^2)$$

Scaling: $\psi(x) = \frac{1}{\sqrt{\theta}}\phi_m(\sqrt{m}x)$ and $\zeta(x) = \frac{1}{2\sqrt{\theta m}}\chi_m(\sqrt{m}x)$

$$\begin{cases} -i\boldsymbol{\sigma}\cdot\nabla\chi_m + 2(S+V+\mu)\phi_m = 0 \\ -i\boldsymbol{\sigma}\cdot\nabla\phi_m = (2m-\mu+S-V)(\chi_m/2m) \\ S+V \simeq -\frac{1}{\theta}\left(\frac{g_\sigma^2}{m_\sigma^2} - \frac{g_\omega^2}{m_\omega^2}\right)|\phi_m|^2 + \frac{1}{4\theta}\left(\frac{g_\sigma^2}{m_\sigma^2} + \frac{g_\omega^2}{m_\omega^2}\right)\frac{|\chi_m|^2}{m} \\ S-V \simeq -\frac{1}{\theta}\left(\frac{g_\sigma^2}{m_\sigma^2} + \frac{g_\omega^2}{m_\omega^2}\right)|\phi_m|^2 + \frac{1}{4\theta}\left(\frac{g_\sigma^2}{m_\sigma^2} - \frac{g_\omega^2}{m_\omega^2}\right)\frac{|\chi_m|^2}{m} \end{cases}$$

Non-relativistic limit: σ model

σ model: $V \equiv 0$, $g_\omega = 0$

Choice of parameters: $\dfrac{g_\sigma^2}{m_\sigma^2} = \kappa\theta \Rightarrow S \simeq -\kappa|\phi_m|^2 + O(1/m)$

$$\begin{cases} -i\boldsymbol{\sigma}\cdot\nabla\chi_m + 2S\phi_m + 2\mu\phi_m = 0 \\ -i\boldsymbol{\sigma}\cdot\nabla\phi_m = \chi_m + O(1/m) \end{cases}$$

In the limit $m \to +\infty$, we recover the NLS equation

$$-\Delta\phi - 2\kappa|\phi|^2\phi + 2\mu\phi = 0 \qquad \text{(NLS*)}$$

Assumption: $\phi = \varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$-\Delta\varphi - 2\kappa|\varphi|^2\varphi + 2\mu\varphi = 0 \qquad \text{(NLS)}$$

Remark: For $\kappa, \mu > 0$, the nonlinear equation NLS has a unique positive solution. It is radial, decreasing, and non-degenerate. Non-degenerate: the kernel of the linearized operator at our solution is trivial, *i.e.* it is given by $\mathrm{span}\{\varphi, \partial_{x_1}\varphi, \partial_{x_2}\varphi, \partial_{x_3}\varphi\}$

- $\mathcal{E}_{\mathrm{NLS}}(\varphi) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3} |\nabla\varphi|^2 - \dfrac{\kappa}{2}\int_{\mathbb{R}^3} |\varphi|^4$ on $H^1(\mathbb{R}^3)$ is unbounded from below
- φ is concentrate at the origin when κ is large

Non-relativistic limit: $\sigma - \omega$ model

Choice of parameters: $\frac{g_\sigma^2}{m_\sigma^2} = \theta m, \quad \frac{g_\sigma^2}{m_\sigma^2} - \frac{g_\omega^2}{m_\omega^2} = \lambda \Rightarrow$

$$S + V \simeq -\frac{\lambda}{\theta}|\phi_m|^2 + \frac{1}{2}|\chi_m|^2 + O(1/m) \quad S - V \simeq -2m|\phi_m|^2 + O(1)$$

$$\begin{cases} -i\boldsymbol{\sigma} \cdot \nabla \chi_m + 2(S+V)\phi_m + 2\mu\phi_m = 0 \\ -i\boldsymbol{\sigma} \cdot \nabla \phi_m = \chi_m \left(1 + \frac{S-V}{2m}\right) + O(1/m) \end{cases}$$

In the limit $m \to +\infty$, we recover the equation

$$-\boldsymbol{\sigma} \cdot \nabla \left(\frac{\boldsymbol{\sigma} \cdot \nabla \phi}{1 - |\phi|^2} \right) + \frac{|\boldsymbol{\sigma} \cdot \nabla \phi|^2}{(1 - |\phi|^2)^2}\phi - \frac{2\lambda}{\theta}|\phi|^2\phi + 2\mu\phi = 0 \qquad \text{(nNLS*)}$$

Assumption: $\phi = \varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$-\nabla \cdot \left(\frac{\nabla \varphi}{1 - |\varphi|^2} \right) + \frac{|\nabla \varphi|^2}{(1 - |\varphi|^2)^2}\varphi - \frac{2\lambda}{\theta}|\varphi|^2\varphi + 2\mu\varphi = 0 \qquad \text{(nNLS)}$$

Nonlinear Schrödinger equation for a nucleon

$$-\nabla \cdot \left(\frac{\nabla \varphi}{1-|\varphi|^2} \right) + \frac{|\nabla \varphi|^2}{(1-|\varphi|^2)^2}\varphi - \frac{2\lambda}{\theta}|\varphi|^2\varphi + 2\mu\varphi = 0 \qquad \text{(nNLS)}$$

Energy functional:

$$\mathcal{E}(\varphi) = \frac{1}{2}\int_{\mathbb{R}^3} \frac{|\nabla \varphi|^2}{(1-|\varphi|^2)_+} - \frac{\lambda}{2\theta}\int_{\mathbb{R}^3} |\varphi|^4$$

NLS with a variable mass

$$X = \left\{ \varphi \in L^2(\mathbb{R}^3), \int_{\mathbb{R}^3} \frac{|\nabla \varphi|^2}{(1-|\varphi|^2)_+} < +\infty \right\} \subset H^1(\mathbb{R}^3)$$

Remark: If $\varphi \in X$ then $\varphi \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ and $|\varphi|^2 \leq 1$ a.e. in $\mathbb{R}^3$

Theorem 1 (Lewin-RN 2014)

For $\frac{\lambda}{2\theta} > \mu > 0$, the nonlinear equation (nNLS) has a unique solution $0 < \varphi < 1$ that tends to 0 at infinity, modulo translations and multiplication by a phase factor. It is radial, decreasing, and non-degenerate.

Remark: The existence part of this theorem is contained in some previous works in collaboration with M. Esteban and L. Le Treust

Perspectives

Perspectives:

- Cauchy problem

$$\begin{cases} i\partial_t \phi = -\boldsymbol{\sigma} \cdot \nabla \left(\dfrac{\boldsymbol{\sigma} \cdot \nabla \phi}{1 - |\phi|^2} \right) + \dfrac{|\boldsymbol{\sigma} \cdot \nabla \phi|^2}{(1 - |\phi|^2)^2} \phi - a|\phi|^2 \phi \\ \phi(0, x) = \phi_0(x) \end{cases}$$

questions: existence of local or global solution, rigorous derivation (work in progress with J. Lampart, L. Le Treust, J. Sabin), stability of stationary solutions

- N-body problem, $N > 1$

$$\mathcal{E}_{a,N}(\Phi) = \frac{1}{2} \sum_{j=1}^{N} \int_{\mathbb{R}^3} \frac{|\nabla \varphi_j|^2}{1 - \rho_\Phi} - \frac{a}{4} \int_{\mathbb{R}^3} \rho_\Phi^2$$

under the constraints $0 < \rho_\Phi < 1$, $\int_{\mathbb{R}^3} \varphi_i^* \varphi_j = \delta_{ij}$ and $\int_{\mathbb{R}^3} \rho_\Phi = N$. Here

$\Phi = (\varphi_1, \ldots, \varphi_n)$ et $\rho_\Phi = \sum_{j=1}^{N} |\varphi_j|^2$.

questions: existence of minimizers in different regimes of parameters (works in progress with J. Lampart and M. Lewin)