Twisted Alexander polynomials and hyperbolic volume for three-manifolds

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December 9, 2019

Discrete Subgroups of Lie Groups
Birs

Main Theorem

- $K \subset S^3$ hyperbolic knot, $S^3 \setminus K = \Gamma \backslash \mathbb{H}^3$, Γ torsion free lattice, $\rho_N \colon \pi_1(S^3 \setminus K) \cong \Gamma \subset \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^{N-1}} \mathrm{SL}_N(\mathbb{C})$
- $\Delta_K^{\rho_N}(t) \in \mathbb{C}[t, t^{-1}]$ Alexander polynomial of K twisted by ρ_N
- It is a Reidemeister torsion: $\Delta_K^{\rho_N}(t) = \tau(S^3 \setminus K, t^{\mathrm{ab}} \otimes \rho_N)^{-1}$ (times $\frac{1}{t-1}$ for N odd), where $\mathrm{ab} \colon \Gamma \twoheadrightarrow \mathbb{Z}$, $t^{\mathrm{ab}} \colon \Gamma \to \mathbb{Z}[t, t^{-1}]$
- When $S^3 \setminus K$ is the mapping torus of a diffeo $\phi \colon \Sigma \to \Sigma$, i.e. when $S^3 \setminus K \cong \Sigma \times [0,1]/(x,1) \sim (\phi(x), 0)$, then $\Delta_K^{\rho_N}(t) = \det(\phi^* - t\,\mathrm{Id})$, where $\phi^* \colon H^1(\Sigma, \rho_N) \to H^1(\Sigma, \rho_N)$

Thm: (BDHP 19) For $\xi \in \mathbb{C}$, $|\xi| = 1$,

$$\lim_{N \to +\infty} \frac{\log |\Delta_K^{\rho_N}(\xi)|}{N^2} = \frac{1}{4\pi} \mathrm{Vol}(S^3 \setminus K)$$

- True for cusped manifolds (with conditions on the "variables" map $\pi_1(M) \twoheadrightarrow \mathbb{Z}^r$ to define $\Delta(t_1, \ldots, t_r)$).

Strategy:

Thm: $K \subset S^3$ hyperbolic knot. For $\xi \in \mathbb{C}$, $|\xi| = 1$,

$$\lim_{N\to+\infty} \frac{\log|\Delta_K^{\rho_N}(\xi)|}{N^2} = \frac{1}{4\pi}\operatorname{Vol}(S^3 \setminus K)$$

- It is a theorem on Reidemeister torsion, as

$$|\Delta_K^{\rho_N}(\xi)| = |\tau(S^3 \setminus K, \xi^{\mathrm{ab}} \otimes \rho_N)^{-1}|,$$

where $\xi^{\mathrm{ab}} \colon \Gamma \to \mathbb{S}^1 \subset \mathbb{C}$ maps $\gamma \mapsto \xi^{\mathrm{ab}(\gamma)}$

Thm (W. Müller 2012):
M^3 *closed* hyperbolic, $\lim_{N\to\infty} \frac{\log|\tau(M^3,\rho_N)|}{N^2} = -\frac{\operatorname{Vol}(M^3)}{4\pi}$

- Strategy:
  - Prove Müller's thm for $\chi \otimes \rho_N$ instead of ρ_N (and M^3 closed), where $\chi \colon \pi_1 M^3 \to \mathbb{S}^1 \subset \mathbb{C}$.
  - Approximate $S^3 \setminus K$ by Dehn fillings.

Analytic torsion

- M closed & smooth, $\rho : \pi_1 M \to \mathrm{SL}_n(\mathbb{R})$, $H^*(M, \rho) = 0$.
  $\Delta^p : \Omega^p(M; \rho) \to \Omega^p(M; \rho)$ Laplacian on E_ρ-valued p-forms.
  $\mathrm{Spec}(\Delta^p)$ is discrete and > 0.

$$\zeta_p(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta^p)} \lambda^{-s} \qquad \text{for } s \in \mathbb{C},\ Re(s) \gg 0$$

  $\zeta_p(s)$ extends holomorphically at $s = 0$.

$$\tau^{anal}(M, \rho) := \exp\left(\frac{1}{2} \sum_p (-1)^{p+1} p\, \zeta_p'(0)\right)$$

Thm: (Cheeger-Müller) Analytic torsion $=$ |Combinatorial torsion|

$$\tau^{anal}(M, \rho) = |\tau^{comb}(M, \rho)|$$

- Proved by J.Cheeger & W.Müller for $\rho : \pi_1 M \to \mathrm{SO}(n)$ (1978)
- Proved by W. Müller for $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ (1993).
- Müller: M^3 closed hyp, $\lim_{N \to \infty} \frac{\log \tau^{anal}(M^3, \rho_N)}{N^2} = -\frac{\mathrm{Vol}(M^3)}{4\pi}$

Ruelle zeta function

- M^3 closed hyperbolic, $\rho : \pi_1(M^3) \to \mathrm{SL}_n(\mathbb{C})$ acyclic.
- Ruelle zeta function:

$$\mathcal{R}_\rho(s) = \prod_{\gamma \in PCG(M^3)} \det(\mathrm{Id} - \rho(\gamma) e^{-s\, l(\gamma)}), \qquad s \in \mathbb{C},\ \mathrm{Re}(s) > 2.$$

$$\begin{aligned} PCG(M^3) &= \{\text{oriented primitive closed geodesics } \gamma \subset M^3\}. \\ &= \{[\gamma] \text{ conjugacy class} \mid 1 \neq \gamma \in \pi_1(M^3) \text{ primitive}\} \\ l(\gamma) &= \text{length}(\gamma) \end{aligned}$$

Thm: $\mathcal{R}_\rho(s)$ extends meromorphically to $\mathbb{C}$ and

$$|\mathcal{R}_\rho(0)| = \tau^{anal}(M^3, \rho)^2,$$

- Proved by D. Fried 1986 for $\rho : \pi_1 M^{2k+1} \to \mathrm{SO}(n)$
- By A. Wotzke in 2008 for $\rho : \pi_1 M^3 \to \mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$

Müller's Thm and its proof

- Müller's Thm: M^3 closed, $\lim_{N\to\infty} \frac{\log|\tau(M^3,\rho_N)|}{N^2} = -\frac{\mathrm{Vol}(M^3)}{4\pi}$
- Tools: Ruelle functions for $s \in \mathbb{C}$, $\mathrm{Re}(s) \gg 0$.
  - $\mathcal{R}_{\rho_N}(s) = \prod_{\gamma\in PCG(M^3)} \det(\mathrm{Id} - \rho_N(\gamma)e^{-sl(\gamma)})$,
  - $R_k(s) = \prod_{\gamma\in PCG(M^3)} (1 - e^{\frac{k}{2}i\theta(\gamma) - sl(\gamma)})$

    where $l(\gamma) + i\theta(\gamma) =$ complex length of γ

$\mathcal{R}_{\rho_N}(s) = \prod_{k=0}^{N-1} R_{N-1-2k}(s - (\frac{N-1}{2} - k))$

- $\mathcal{R}_{\rho_N}(0) = |\tau(M^3,\rho_N)|^2$ and $R_k(s) = e^{\frac{4\mathrm{vol}(M)s}{\pi}} R_{-k}(-s)$

$\Rightarrow$ Müller's formula for $N = 2m$:

$$\log \frac{|\tau(M^3,\rho_{2m})|}{|\tau(M^3,\rho_4)|} = -\frac{1}{\pi}\mathrm{Vol}(M^3)(m^2-4) + \sum_{k=2}^{m-1} \log|R_{-2k-1}(k+\tfrac{1}{2})|$$

- and $\sum_{k=2}^{m-1} \log|R_{-2k-1}(k+\frac{1}{2})| < C$ uniformly on $m = N/2$.

Claim: This formula holds also for $\chi \otimes \rho_N$, where $\chi\colon \pi_1 M^3 \to \mathbb{S}^1 \subset \mathbb{C}$

Next: approximate $S^3 \setminus K$ by closed manifolds (Dehn fillings).

Approximate by Dehn fillings $K_{p/q}$

- $K_{p/q} = S^3 \setminus \mathcal{N}(K) \cup_\varphi D^2 \times S^1$, with
  $\varphi(\partial D^2 \times *) = p\ \mathrm{meridian} + q\ \mathrm{longitude}$

Thm (Thurston) $K_{p/q}$ is hyperbolic for almost every $p/q \in \mathbb{Q} \cup \{\infty\}$ and $\lim_{p^2+q^2 \to \infty} K_{p/q} = S^3 \setminus K$ for the geometric topology.

In particular $\mathrm{Vol}(K_{p/q}) \to \mathrm{Vol}(S^3 \setminus K)$

- The thick part of $K_{p/q}$ converges to the thick part of $S^3 \setminus K$
- The soul of $D^2 \times S^1$ is a geodesic with length $\to 0$ and the Margulis tube around this short geodesic converges to a cusp

Lemma For any $C \geq 1$

$$\{\gamma \in PCG(K_{p/q}) \mid \tfrac{1}{C} \leq l(\gamma) \leq C\} \to \{\gamma \in PCG(S^3 \setminus K) \mid l(\gamma) \leq C\}$$

as $p^2 + q^2 \to \infty$, and the complex lengths converge.

Limit of Müller's formula as $p^2 + q^2 \to \infty$

- When $\xi = e^{2\pi i r/s}$, chose sequences of Dehn fillings $K_{p/q}$ with $p \in s\mathbb{Z}$ so that $\xi^{\mathrm{ab}} \colon \pi_1(S^3 \setminus K) \to \mathbb{S}^1$ factors through $\pi_1 K_{p/q}$.
- Using geometric convergence $K_{p/q} \to S^3 \setminus K$ and dealing with short geodesics, the limit of Müller's formula on $K_{p/q}$ yields

$$\log \left| \frac{\Delta_K^{\rho_{2m}}(\xi)}{\Delta_K^{\rho_4}(\xi)} \right| = \frac{1}{\pi} \mathrm{Vol}(S^3 \setminus K)(m^2 - 4) - \sum_{k=2}^{m-1} \log |R_{\xi,-2k-1}(k + \tfrac{1}{2})|$$

for $\xi \in e^{2\pi i \mathbb{Q}}$,
where $R_{\xi,-2k-1}(k + \frac{1}{2}) = \prod_{\gamma} (1 - \xi^{\mathrm{ab}(\gamma)} e^{-(2k+1)(l(\gamma) + i\theta(\gamma))/2})$

- To prove the theorem for *any* $\xi \in \mathbb{S}^1 \subset \mathbb{C}$, prove:
  - $\sum_{k=2}^{m-1} \log |R_{\xi,-2k-1}(k + \frac{1}{2})|$ is unif. bounded & cont. on $\xi \in \mathbb{S}^1$
  - $\Delta_K^{\rho_{2m}}(\xi) \neq 0$ for any $\xi \in \mathbb{S}^1 \subset \mathbb{C}$ ($H^*(S^3 \setminus K, \xi^{\mathrm{ab}} \otimes \rho_N) = 0$).

Then the formula holds for any $\xi \in \mathbb{S}^1$ by continuity.