Analytical solutions and attractors of higher-order viscous hydrodynamics

Amaresh Jaiswal

National Institute of Science Education and Research, Bhubaneswar, India

Theoretical Foundations of Relativistic Hydrodynamics

Banff International Research Station, Canada

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Collaborators: S. Jaiswal, C. Chattopadhyay, S. Pal & U. Heinz

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Overview

- Attractor in minimal causal Maxwell-Cattaneo theory.
- Higher-order hydrodynamic theories: MIS, DNMR and Third-order.
- Setup: Conformal system + Bjorken flow.
- Fixed points, Lyapunov exponents and attractors.
- Approximate analytical solutions.
- Attractor and Lyapunov exponent from analytical solutions.
- Convergence of IC in small and large Knudsen number regime.

Relativistic hydrodynamics: Navier-Stokes

- Degrees of freedom: Local energy density (ϵ), thermodynamic pressure (P), hydrodynamic four velocity (u^μ).

- General form of energy-momentum tensor:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}.$$

$$\pi^{\mu\nu} = 2\,\eta\left[\frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha\right].$$

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu, \qquad \nabla^\alpha \equiv \Delta^{\alpha\beta}\partial_\beta.$$

- $u_\nu\partial_\mu T^{\mu\nu} = 0 \Rightarrow$ Continuity equation.

- $\Delta^\alpha_\nu\partial_\mu T^{\mu\nu} = 0 \Rightarrow$ Navier-Stokes equation.

- Relativistic Navier-Stokes is acausal theory!

Maxwell-Cattaneo law: Minimal causal theory

J. C. Maxwell, Phil. Trans. R. Soc. 157:49 (1867),
C. Cattaneo. Sulla conduzione del calore. Atti Sem. Mat. Fis. Univ. Modena, 3:3, (1948)

- Simplest way to restore causality: "Maxwell-Cattaneo" law−

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle}.$$

Dissipative forces relax to their Navier-Stokes values in some finite relaxation time τ_π: Restores causality.

Attractor in Maxwell-Cattaneo

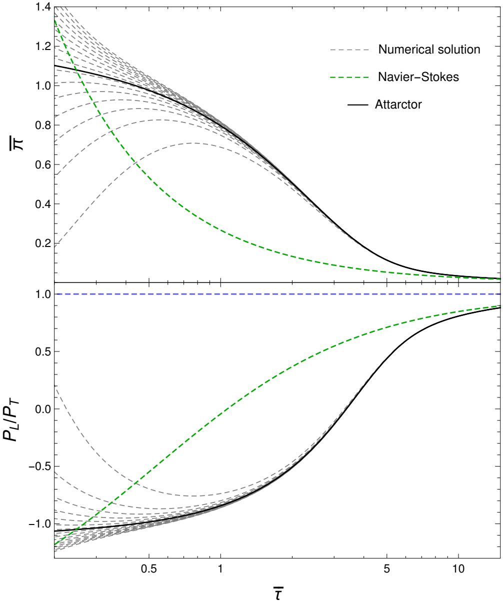
- Consider conformal system + Bjorken flow.
- Energy conservation and shear evolution equation:

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\frac{4}{3}\epsilon - \pi\right), \qquad \frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \frac{16}{45}\frac{\epsilon}{\tau} .$$

- Can be decoupled. Normalized shear ($\bar{\pi}$) evolution equation:

$$\left(\frac{\bar{\pi}+2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} + \bar{\pi} = \frac{1}{\bar{\tau}}\left[\frac{4}{15} + \frac{4}{3}\bar{\pi} - \frac{4}{3}\bar{\pi}^2\right], \qquad \bar{\pi} \equiv \frac{\pi}{\epsilon + P}, \quad \bar{\tau} \equiv \frac{\tau}{\tau_\pi} .$$

Attractor in Maxwell-Cattaneo theory



$$\bar{\pi} \equiv \frac{1}{\text{Reynolds No.}} \equiv \frac{\pi}{\epsilon + P}$$

$$\bar{\tau} \equiv \frac{1}{\text{Knudsen No.}} \equiv \frac{\tau}{\tau_\pi}$$

$$\frac{P_L}{P_T} = \frac{1 - 4\bar{\pi}}{1 + 2\bar{\pi}}$$

Attractor exists for all causal hydrodynamic theories!

Hydrodynamics from kinetic theory

- Hydrodynamic theories can be derived from kinetic theory assuming system to be close to thermal equilibrium, $f = f_0 + \delta f$.

$$T^{\mu\nu}(x) = \int dp\; p^\mu p^\nu\, f\,(x,p), \qquad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f.$$

- Boltzmann equation in the relxn. time approx. is solved iteratively:

$$p^\mu \partial_\mu f = -\frac{u\cdot p}{\tau_R}(f - f_0) \;\Rightarrow\; f = f_0 - (\tau_R/u\cdot p)\, p^\mu \partial_\mu f.$$

- Expand f about its equilibrium value: $f = f_0 + \delta f^{\,(1)} + \delta f^{\,(2)} + \cdots$,

$$\begin{aligned}
\delta f^{\,(1)} &= -\frac{\tau_R}{u\cdot p}\, p^\mu \partial_\mu f_0, \\
\delta f^{\,(2)} &= \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu\left(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\right), \\
\delta f^{\,(3)} &= \ldots
\end{aligned}$$

Second-order hydrodynamics

- Keep all terms till second order for conformal system. Substituting $\delta f = \delta f^{\ (1)} + \delta f^{\ (2)}$ [AJ, PRC 87, 051901 (2013)]:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma},$$

where $\beta_\pi \equiv \frac{\eta}{\tau_\pi} = \frac{4P}{5}$. DNMR theory

[G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D85, 114047 (2012)]

- For minimal causal conformally symmetric systems:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta.$$

We will call it "MIS" theory.

Close variant of : I. Muller, Z. Phys. 198, 329 (1967)

W. Israel and J. M. Stewart, Annals Phys. 118, 341 (1979)

Third-order hydrodynamics

Third-order equation for shear stress tensor [AJ, PRC 88, 021903 (2013)]

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{63}\pi^{\mu\nu}\theta^2$$
$$+\tau_\pi\left[\frac{50}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\sigma_{\rho\gamma} - \frac{76}{245}\pi^{\mu\nu}\sigma^{\rho\gamma}\sigma_{\rho\gamma} - \frac{44}{49}\pi^{\rho\langle\mu}\sigma^{\nu\rangle\gamma}\sigma_{\rho\gamma}\right.$$
$$\left.-\frac{2}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{2}{7}\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} + \frac{26}{21}\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta - \frac{2}{3}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}\theta\right]$$
$$-\frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right)$$
$$-\frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right).$$

- Causal and leads to improved accuracy compared to second-order.
- Neccessary for incorporation of colored noise in fluctuating hydro evolution [J. Kapusta and C. Young, Phys. Rev. C 90, 044902 (2014)].

Bjorken flow [J. D. Bjorken, PRD 27, 140 (1983)]

- For boost-invariant longitudinal expansion, $v^z = \frac{z}{t}$, $v^x = v^y = 0$.
- Milne coordinate system: proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\eta_s = \tanh^{-1}(z/t)$.

Hydrodynamic equations for Bjorken for MIS, DNMR and Third-order theories can be brought into the generic form:

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\frac{4}{3}\epsilon - \pi\right), \quad \frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \frac{1}{\tau}\left[\frac{4}{3}\beta_\pi - \left(\lambda + \frac{4}{3}\right)\pi - \chi\frac{\pi^2}{\beta_\pi}\right],$$

where $\beta_\pi \equiv \frac{\eta}{\tau_\pi} = \frac{4P}{5}$ and $\tau_\pi = 5\bar{\eta}/T$.

Theory	β_π	a	λ	χ	γ
MIS	$4P/5$	4/15	0	0	4/3
DNMR	$4P/5$	4/15	10/21	0	4/3
Third-order	$4P/5$	4/15	10/21	72/245	412/147

The coefficients β_π, a, λ, χ, and γ for MIS, DNMR and Third-order.

Bjorken equations: Lyapunov exponent

- Bjorken equations in terms of dimensionless parameters, propertime variable $\bar{\tau} \equiv \tau/\tau_\pi$ and normalized shear $\bar{\pi} \equiv \pi/(\epsilon + P)$:

$$\frac{d\bar{\tau}}{d\tau} = \left(\frac{\bar{\pi}+2}{3}\right)\frac{\bar{\tau}}{\tau}, \qquad \left(\frac{\bar{\pi}+2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$$

- Series expansion in powers of $1/\bar{\tau}$,

$$\bar{\pi}(\bar{\tau}) = \sum_{n=1}^{\infty}\frac{c_n}{\bar{\tau}^n} = \frac{a}{\bar{\tau}} + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right).$$

- Linear perturbation around this solution:

$$\delta\bar{\pi}(\bar{\tau}) \sim \bar{\tau}^{\frac{3}{4}(a-2\lambda)}\exp\left(-\frac{3}{2}\bar{\tau}\right)\left[1 + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right)\right].$$

Lyapunov exponent: $\Lambda = -3/2$.

M. P. Heller and M. Spaliski, Phys.Rev. Lett. 115 (2015) 072501 [1503.07514]
G. Basar and G. V. Dunne, Phys. Rev. D92 (2015) 125011 [1509.05046]
A. Behtash, S. Kamata, M. Martinez and H. Shi, Phys. Rev. D99 (2019) 116012

“Effective” MIS

- Bjorken equation in case of MIS for inverse Reynolds number as a function of inverse Knudsen number:

$$\left(\frac{\bar{\pi}+2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(a - \frac{4}{3}\bar{\pi}^2\right).$$

- Series expansion in powers of $1/\bar{\tau}$,

$$\bar{\pi}(\bar{\tau}) = \sum_{n=1}^{\infty}\frac{c_n}{\bar{\tau}^n}, \quad c_n = a\,\delta_{n,1} + \frac{2}{3}(n-1)c_{n-1} + \sum_{m=1}^{n}\frac{m-5}{3}\,c_{n-m}\,c_{m-1}.$$

- At late times $\bar{\pi} \ll 1$, hence $\bar{\pi}^2 \ll \bar{\pi}$, and the nonlinear terms can be ignored. Dominated by factorial growth of coefficients.
- “Effective” MIS equation and solution:

$$\frac{2}{3}\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{a}{\bar{\tau}} \quad \Rightarrow \quad \bar{\pi} = \alpha\, e^{-\frac{3}{2}\bar{\tau}} + \frac{3a}{2}\, e^{-\frac{3}{2}\bar{\tau}}\,\mathrm{Ei}\left[\frac{3\bar{\tau}}{2}\right]$$

- Separation between two solutions for $\bar{\pi}$ with different initial conditions is damped exponentially: $\dfrac{\partial\bar{\pi}}{\partial\alpha} \sim \exp\left(-\frac{3}{2}\bar{\tau}\right).$

Bjorken equations: Fixed points

- Hydrodynamic equations in terms of temperature $T(\tau)$ and $\bar{\pi}(\tau)$:

$$\frac{dT}{d\tau} = \frac{T}{3\,\tau}\,(\bar{\pi} - 1)\,, \qquad \frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}\,T}{5\bar{\eta}} + \frac{1}{\tau}\,(a - \lambda\bar{\pi} - \gamma\bar{\pi}).$$

- Both derivatives should vanish at the fixed points. This conditions is satisfied at:

$$(0, \bar{\pi}_+, \tau), \quad (0, \bar{\pi}_-, \tau), \quad \left(-\frac{5\bar{\eta}}{\tau}\,(\lambda + \gamma - a)\,, 1, \tau\right).$$

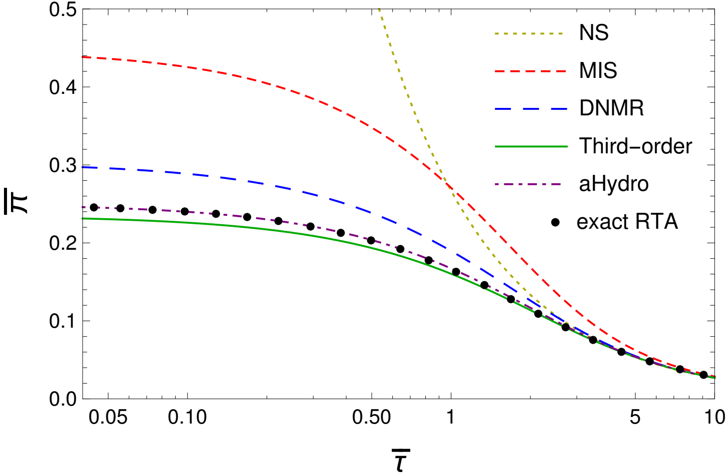
Notation: $(T, \bar{\pi}, \tau)$, $\qquad \bar{\pi}_\pm \equiv \frac{-\lambda \pm \sqrt{4a\gamma + \lambda^2}}{2\gamma}$.

- First fixed point is the stable fixed point (attractor). The second fixed point is unstable (repulsor).

- Third fixed point (red) lies in unphysical region (-ve temperature).

Attractors for different theories

[S. Jaiswal, C. Chattopadhyay, AJ, S. Pal, and U. Heinz, Phys. Rev. C100, 034901 (2019)]



Numerical attractors are obtained following the prescription−
M. P. Heller and M. Spaliski, Phys.Rev. Lett. 115 (2015) 072501 [1503.07514]

Exact differential equation:

$$\left(\frac{\bar{\pi}+2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right)$$

Has the form of an Abel differential equation of the second kind for which, to the best of our knowledge, an analytical solution does not exist.

Note: DE has same form as in Maxwell-Cattaneo case! Higher-order theories only have effect on the coefficients (for conformal-Bjorken).

Approximate solutions ⇒

Analytical solution assuming const. relaxation time

G. S. Denicol and J. Noronha, PRD 97, 056021 (2018).

- Bjorken equations can also be written as,

$$\frac{1}{\epsilon\tau^{4/3}}\frac{d(\epsilon\tau^{4/3})}{d\tau} = \frac{4}{3}\frac{\bar{\pi}}{\tau}, \qquad \frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$$

- Assume a constant relaxation time: $\tau_\pi(\tau) = \mathrm{const.}$

Introduces new length scale τ_π in addition to $1/T$. Consequences on Lyapunov exponent.

- Equation in terms of $\{\bar{\pi}, \bar{\tau}\}$:

$$\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$$

Riccati equation. Solution exists.

Analytical solution approximating relaxation time from ideal hydrodynamics:

- Bjorken equations:

$$\frac{1}{\epsilon\tau^{4/3}}\frac{d(\epsilon\tau^{4/3})}{d\tau} = \frac{4}{3}\frac{\bar\pi}{\tau}, \qquad \frac{d\bar\pi}{d\tau} = -\frac{\bar\pi}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar\pi - \gamma\bar\pi^2\right).$$

- For conformal system: $\tau_\pi \propto 1/T$.

- Temperature from ideal fluid law: $T_{\rm id} = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3}$

$$\implies \tau_\pi(\tau) = b\tau^{1/3}, \qquad b \equiv \frac{5\bar\eta}{T_0\tau_0^{1/3}} = \text{const}.$$

- Equation reduces to Riccati equation:

$$\frac{2}{3}\frac{d\bar\pi}{d\bar\tau} = -\bar\pi + \frac{1}{\bar\tau}\left(a - \lambda\bar\pi - \gamma\bar\pi^2\right).$$

Analytical solution approximating relaxation time from Navier-Stokes evolution:

- Bjorken equations:

$$\frac{1}{\epsilon\tau^{4/3}}\frac{d(\epsilon\tau^{4/3})}{d\tau} = \frac{4}{3}\frac{\bar{\pi}}{\tau}, \qquad \frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$$

- Temperature from NS: $T_{\rm NS} = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3}\left[1 + \frac{2\bar{\eta}}{3\tau_0 T_0}\left\{1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right\}\right]$

$$\Longrightarrow\ \tau_\pi(\tau) = \frac{\tau^{1/3}}{d - \frac{2}{15}\tau^{-2/3}}, \qquad d \equiv \left(\frac{T_0\tau_0}{5\bar{\eta}} + \frac{2}{15}\right)\tau_0^{-2/3} = \text{const.}$$

- Reduces to Riccati equation:

$$\left(\frac{a/\bar{\tau} + 2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right).$$

General solutions for all three approximations

[S.Jaiswal, C. Chattopadhyay, AJ, S. Pal, and U. Heinz, Phys. Rev. C100, 034901 (2019)]

Approximate analytical solutions for all three cases in terms of Whittaker functions $M_{k,m}(\bar\tau)$ and $W_{k,m}(\bar\tau)$:

$$\bar\pi(\bar\tau) = \frac{(k+m+\frac{1}{2})M_{k+1,m}(w) - \alpha\, W_{k+1,m}(w)}{\gamma|\Lambda|\left[M_{k,m}(w) + \alpha\, W_{k,m}(w)\right]},$$

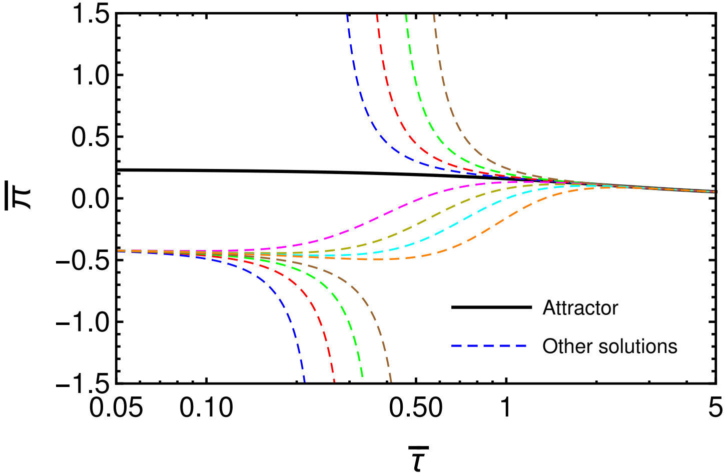
$$\epsilon(\bar\tau) = \epsilon_0 \left(\frac{w_0}{w}\right)^{\frac{4}{3}\left(|\Lambda| - \frac{k}{\gamma}\right)} e^{-\frac{2}{3\gamma}(w-w_0)} \left(\frac{M_{k,m}(w) + \alpha\, W_{k,m}(w)}{M_{k,m}(w_0) + \alpha\, W_{k,m}(w_0)}\right)^{\frac{4}{3\gamma}}.$$

τ_π	w	Λ	k	m
const.	$\bar\tau$	-1	$-\frac{1}{2}(\lambda+1)$	$\frac{1}{2}\sqrt{4a\gamma+\lambda^2}$
$\sim 1/T_{\rm id}$	$\frac{3}{2}\bar\tau$	$-\frac{3}{2}$	$-\frac{1}{4}(3\lambda+2)$	$\frac{3}{4}\sqrt{4a\gamma+\lambda^2}$
$\sim 1/T_{\rm NS}$	$\frac{3}{2}\left(\bar\tau+\frac{a}{2}\right)$	$-\frac{3}{2}$	$-\frac{1}{4}\left(3\left(\lambda-\frac{a}{2}\right)+2\right)$	$\frac{3}{4}\sqrt{4a\gamma+\left(\lambda-\frac{a}{2}\right)^2}$

Arguments and parameters for the obtained analytical solutions.

α encodes the initial condition $\bar\pi_0$.

Attractor and repulsor behavior at $\bar{\tau} \to 0$



At $\bar{\tau} \to 0$, all the evolution trajectories except the attractor converge to the repulsor point.

Analytical attractors

[S.Jaiswal, C. Chattopadhyay, A. J., S. Pal, and U. Heinz, Phys. Rev. C100, 034901 (2019)]

- **Uniquely determining attractor:**
  We propose the quantity—

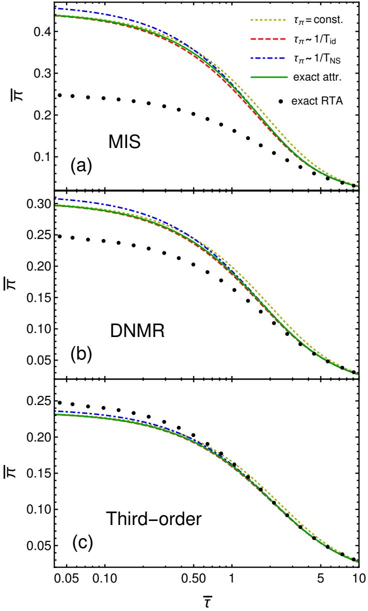
$$\psi(\alpha_0) \equiv \lim_{\bar{\tau}\to\bar{\tau}_0} \frac{\partial \bar{\pi}}{\partial \alpha}\bigg|_{\alpha=\alpha_0}$$

  diverges at α_0 which corresponds to attractor. $\bar{\tau}_0$ slice contains fixed points. $\alpha_0 = 0$ for cases studied here.

- **Attractor solution:**

$$\bar{\pi}_{\rm attr}(\bar{\tau}) = \frac{k+m+\frac{1}{2}}{\gamma|\Lambda|}\,\frac{M_{k+1,m}(w)}{M_{k,m}(w)}.$$

  Independent of initial conditions α!



Lyapunov exponent from analytical solutions

[S.Jaiswal, C. Chattopadhyay, AJ, S. Pal, and U. Heinz, Phys. Rev. C100, 034901 (2019)]

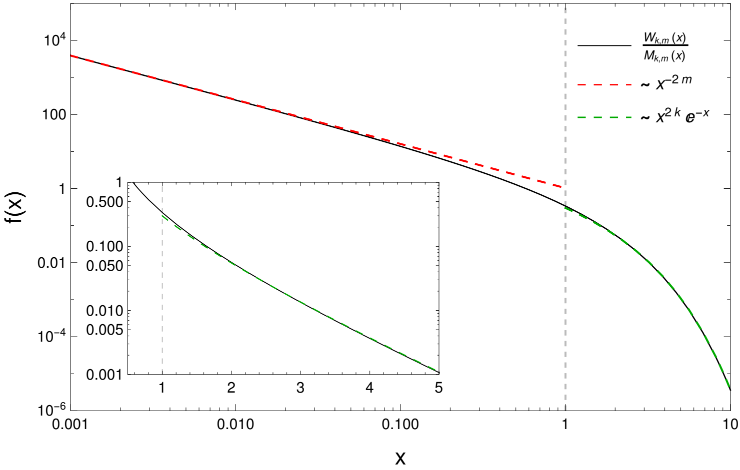
Lyapunov exponent (Λ) can be extracted from the analytical solutions−

$$\Lambda = \lim_{\bar{\tau}\to\infty} \frac{\partial}{\partial\bar{\tau}} \left[\ln\left(\frac{\partial\bar{\pi}}{\partial\alpha} \right) \right].$$

For constant relaxation time approximation, $\Lambda = -1$. Consequence of introducing new length scale τ_π.

For the other two cases (τ_π from ideal and NS), $\Lambda = -\frac{3}{2}$.

Power law and exponential decay of initial conditions



$\delta\bar{\pi} \propto \frac{W_{k,m}(x)}{M_{k,m}(x)}$.

Power-law decay ($\delta\bar{\pi} \approx \bar{\tau}^{-2m}$) in large Knudsen number regime . [A. Kurkela, U. A. Wiedemann, and B. Wu, (2019), 1907.08101].

Exponential decay ($\delta\bar{\pi} \approx \bar{\tau}^{2k} e^{-|\Lambda|\bar{\tau}}$) for small Knudsen numbers.

Summary

- Existence of attractor in Minimal causal theory.
- Comparison of attractors for various hydrodynamic theories.
- Analytical solutions for different hydrodynamic theories for Bjorken expansion in different approximations.
- Uniquely determining attractor from obtained analytical solutions.
- Lyapunov exponents from analytical solutions.
- Early and late "time" behavior of initial conditions.

Thank You!