# *A quantum information perspective on relativistic fluid dynamics and quantum fields out-of-equilibrium*

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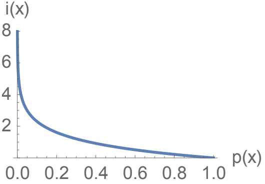
# *Entropy and information*

[Claude Shannon (1948)]

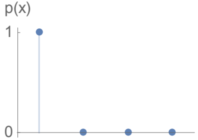
- consider a random variable $x$ with probability distribution $p(x)$
- information content or "surprise" associated with outcome $x$

$$i(x) = -\ln p(x)$$

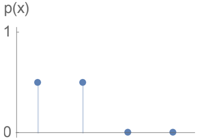



- Entropy is expectation value of information content

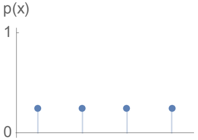
$$S = \langle i(x) \rangle = -\sum_x p(x) \ln p(x)$$




$S = 0$




$S = \ln(2)$




$S = 2\ln(2)$

# *Entropy in quantum theory*

[John von Neumann (1932)]

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- based on the quantum density operator $\rho$
- for pure states $\rho = |\psi\rangle\langle\psi|$ one has $S = 0$
- for mixed states $\rho = \sum_j p_j |j\rangle\langle j|$ one has $S = -\sum_j p_j \ln p_j > 0$
- unitary time evolution conserves entropy

$$-\mathsf{Tr}\{(U\rho U^\dagger)\ln(U\rho U^\dagger)\} = -\mathsf{Tr}\{\rho \ln \rho\} \qquad \rightarrow \qquad S = \mathsf{const.}$$

- quantum information is globally conserved

# *Relativistic fluid dynamics*

- approximate description of quantum field dynamics
- local dissipation = local entropy production

$$-\nabla_\mu s^\mu(x) > 0$$

- e. g. in Navier-Stokes approximation

$$-\nabla_\mu s^\mu = \frac{1}{T}\left[2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2\right]$$

- crucial difference to quantum field theory: entropy not conserved

# *What is an entropy current?*

- *can not* be density of global von-Neumann entropy for closed system

$$\int_\Sigma d\Sigma_\mu \, s^\mu(x) \neq -\mathsf{Tr}\left\{\rho \ln \rho\right\}$$

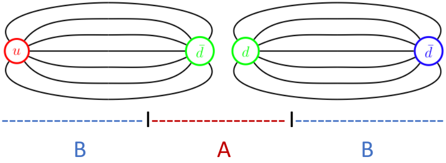
- kinetic theory for weakly coupled (quasi-) particles [Boltzmann (1890)]

$$s^\mu(x) = -\int \frac{d^3p}{p^0} \left\{p^\mu f(x,p) \ln f(x,p)\right\}$$

- molecular chaos: keep only single particle distribution $f(x,p)$
- how to go beyond weak coupling / quasiparticles?
- aim: local notion of entropy in QFT

# *Entropy and entanglement*

- consider a split of a quantum system into two $A + B$




- reduced density operator for system $A$

$$\rho_A = \mathrm{Tr}_B\{\rho\}$$

- entropy associated with subsystem A

$$S_A = -\mathrm{Tr}_A\{\rho_A \ln \rho_A\}$$

- pure **product** state $\rho = \rho_A \otimes \rho_B$ leads to $S_A = 0$
- pure **entangled** state $\rho \neq \rho_A \otimes \rho_B$ leads to $S_A > 0$
- $S_A$ is called **entanglement entropy**

## *Classical statistics*

- consider system of two random variables $x$ and $y$
- joint probability $p(x, y)$ , joint entropy

$$S = -\sum_{x,y} p(x, y) \ln p(x, y)$$

- reduced or marginal probability $p(x) = \sum_y p(x, y)$
- reduced or marginal entropy

$$S_x = -\sum_{x} p(x) \ln p(x)$$

- one can prove: **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state $S = 0$ is also **locally pure** $S_x = 0$

# *Quantum statistics*

- consider system with two subsystems $A$ and $B$
- combined state $\rho$ , combined or full entropy

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- reduced density matrix $\rho_A = \mathsf{Tr}_B\{\rho\}$
- reduced or entanglement entropy

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- for quantum systems **entanglement makes a difference**

$$S \not\geq S_A$$

- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive**!
- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$

# *Entanglement entropy in quantum field theory*

- entanglement entropy of region $A$ is a local notion of entropy

$$S_A = -\mathsf{tr}_A\{\rho_A \ln \rho_A\} \qquad \rho_A = \mathsf{tr}_B\{\rho\}$$

- however, it is infinite already in vacuum state

$$S_A = \frac{\mathsf{const}}{\epsilon^{d-2}} \int_{\partial A} d^{d-2}\sigma\sqrt{h} + \dots$$

- UV divergence proportional to entangling surface
- quantum fields are very strongly entangled already in vacuum
- Theorem [Reeh & Schlieder (1961)]: local operators in region $A$ can create all particle states

# *Relative entropy*

- **relative entropy** of two density matrices

$$S(\rho|\sigma) = \mathsf{tr}\,\{\rho\,(\ln\rho - \ln\sigma)\}$$

- measures how well state $\rho$ can be distinguished from a model $\sigma$
- Gibbs inequality: $S(\rho|\sigma) \geq 0$
- $S(\rho|\sigma) = 0$ if and only if $\rho = \sigma$
- quantum generalization of Kullback-Leibler divergence

# *Relative entanglement entropy*

- consider now reduced density matrices

$$\rho_A = \mathsf{Tr}_B\{\rho\}, \qquad \sigma_A = \mathsf{Tr}_B\{\sigma\}$$

- define **relative entanglement entropy**

$$S_A(\rho|\sigma) = \mathsf{Tr}\left\{\rho_A\left(\ln \rho_A - \ln \sigma_A\right)\right\}$$

- measures how well $\rho$ is represented by $\sigma$ locally in region $A$
- UV divergences cancel: contains real physics information
- well defined in quantum field theory [Araki (1977), see also Witten (2018)]

# *An approximate local description*

- consider non-equilibrium situation with
  - true density matrix $\rho$
  - local equilibrium approximation

$$\sigma = \frac{1}{Z} e^{-\int d\Sigma_\mu \{\beta_\nu(x) T^{\mu\nu} + \alpha(x) N^\mu\}}$$

- reduced density matrices $\rho_A = \mathrm{Tr}_B\{\rho\}$ and $\sigma_A = \mathrm{Tr}_B\{\sigma\}$
- $\sigma$ is very good model for $\rho$ in region $A$ when

$$S_A = \mathrm{Tr}_A\{\rho_A(\ln \rho_A - \ln \sigma_A)\} \to 0$$

- does *not* imply that globally $\rho = \sigma$

# *Monotonicity of relative entropy*

- monotonicity of relative entropy

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) \leq S(\rho|\sigma)$$

  with $\mathcal{N}$ completely positive, trace-preserving map
- $\mathcal{N}$ unitary evolution

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) = S(\rho|\sigma)$$

- $\mathcal{N}$ open system evolution with generation of entanglement to environment

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) < S(\rho|\sigma)$$

# *Local form of second law*

- for small volume $A \to 0$ (hypothesis)

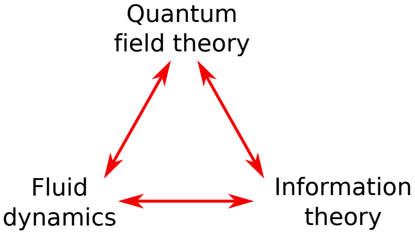
$$S_A(\rho|\sigma) = \int_A d\Sigma_\mu s^\mu(\rho|\sigma)$$

- local form of second law of thermodynamics

$$-\nabla_\mu s^\mu(\rho|\sigma) \leq 0$$

- **relative entanglement entropy between $\rho$ and thermal state $\sigma$ is non-increasing**

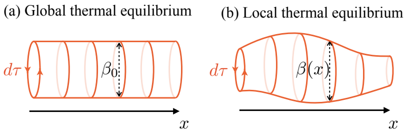
# *Quantum field dynamics*




- quantum information is spread
- locally, quantum state approaches mixed state form
- full loss of *local* quantum information = *local* thermalization
- fluid dynamics + coherent quantum fields with local dissipation

# *Local equilibrium & partition function*

[Floerchinger, JHEP 1609, 099 (2016)]




- local equilibrium with $T(x)$ and $u^\mu(x)$

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

- represent partition function as functional integral with periodicity

$$\phi(x^\mu - i\beta^\mu(x)) = \pm\phi(x^\mu)$$

- partition function $Z[J]$, Schwinger functional $W[J]$ in Euclidean

$$Z[J] = e^{W_E[J]} = \int D\phi \, e^{-S_E[\phi] + \int_x J\phi}$$

# *Entropy production*

[Floerchinger, JHEP 1609, 099 (2016)]

- variational principle with effective dissipation from analytic continuation
- analysis of general covariance leads to entropy current and local entropy production

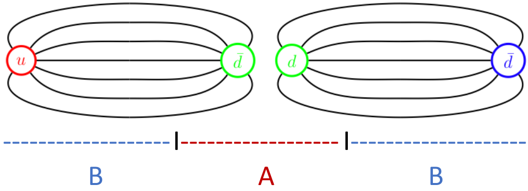
$$\nabla_\mu s^\mu = \frac{1}{\sqrt{g}} \frac{\delta \Gamma_D}{\delta \Phi_a}\Big|_{\text{ret}} \beta^\lambda \partial_\lambda \Phi_a + \beta_\mu \nabla_\nu \left( -\frac{2}{\sqrt{g}} \frac{\delta \Gamma_D}{\delta g_{\mu\nu}}\Big|_{\text{ret}} \right)$$

- can likely be understood as entanglement generation

# *Thermalization beyond collisions*

- quantum fields can be locally thermal without collisions
- horizons: black holes, de-Sitter space
- space-time dynamics of entanglement

# *Entanglement, QCD strings and thermalization*




- hadronization in Lund string model (e. g. PYTHIA)
- reduced density matrix for region $A$

$$\rho_A = \mathsf{Tr}_B\{\rho\}$$

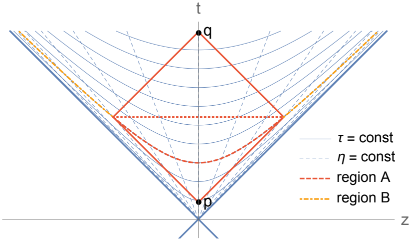
- **entanglement entropy**

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- could this lead to thermal-like effects?

# *Local density matrix and temperature in expanding string*

[Berges, Floerchinger, Venugopalan, *Thermal excitation spectrum from entanglement in an expanding quantum string*, PLB778, 442 (2018)]




- Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- **local density matrix thermal at early times as result of entanglement**

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

- Hawking-Unruh temperature in Rindler space $T(x) = \frac{\hbar c}{2\pi x}$

# *Conclusions*

- new perspectives on relativistic fluids dynamics from quantum information theory
- relative entanglement entropy useful to describe local thermalization
- quantum field theoretic description of relativistic fluid dynamics with two density matrices
- true density matrix $\rho$ evolves unitary
- fluid model $\sigma$ agrees locally but evolves non-unitary
- local thermalization without collisions possible

## *Backup*

# *One-particle irreducible or quantum effective action*

- in Euclidean domain $\Gamma[\phi]$ defined by Legendre transform

$$\Gamma_E[\Phi] = \int_x J_a(x)\Phi_a(x) - W_E[J]$$

with expectation values

$$\Phi_a(x) = \frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J_a(x)}W_E[J]$$

- **Euclidean** field equation

$$\frac{\delta}{\delta\Phi_a(x)}\Gamma_E[\Phi] = \sqrt{g}(x)\, J_a(x)$$

resembles classical equation of motion for $J = 0$

- need **analytic continuation** to obtain a viable equation of motion

# *Analytic continuation*

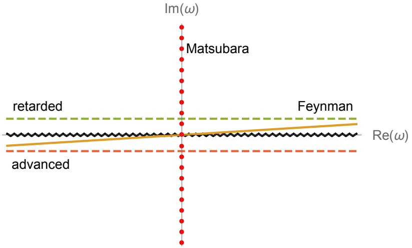
- for homogeneous background field and in global equilibrium

$$\frac{\delta^2}{\delta J_a(-p)\delta J_b(q)} W_E[J] = G_{ab}(p)\ (2\pi)^4 \delta^{(4)}(p-q)$$
$$\frac{\delta^2}{\delta \Phi_a(-p)\delta \Phi_b(q)} \Gamma_E[\Phi] = P_{ab}(p)\ (2\pi)^4 \delta^{(4)}(p-q)$$

- from definition of effective action

$$\sum_b G_{ab}(p) P_{bc}(p) = \delta_{ac}$$

- correlation functions can be analytically continued in $\omega = -u^\mu p_\mu$
- branch cut on real frequency axis $\omega \in \mathbb{R}$



# *Variational principle with effective dissipation*

[Floerchinger, JHEP 1609, 099 (2016)]

- decompose inverse two-point function

$$P_{ab}(p) = P_{1,ab}(p) - i s_{\mathsf{I}}(-u^\mu p_\mu)\, P_{2,ab}(p)$$

with $s_{\mathsf{I}}(\omega) = \text{sign}(\text{Im}\ \omega)$

- in position space, replace

$$\begin{aligned} & s_{\mathsf{I}}\left(-u^\mu p_\mu\right) = \textsf{sign}\left(\textsf{Im}(-u^\mu p_\mu)\right) \\ & \to \textsf{sign}\left(\textsf{Im}\left(i u^\mu \tfrac{\partial}{\partial x^\mu}\right)\right) = \textsf{sign}\left(\textsf{Re}\left(u^\mu \tfrac{\partial}{\partial x^\mu}\right)\right) = s_{\mathsf{R}}\left(u^\mu \tfrac{\partial}{\partial x^\mu}\right) \end{aligned}$$

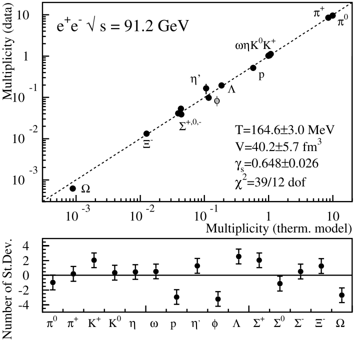
- this symbol appears also in $\Gamma[\Phi]$
- **real and causal** field equations follow from

$$\left.\frac{\delta \Gamma[\Phi]}{\delta \Phi_a(x)}\right|_{\text{ret}} = 0$$

with certain algebraic rules for $s_{\mathsf{R}}\left(u^\mu \frac{\partial}{\partial x^\mu}\right) \to \pm 1$

# *The thermal model puzzle*

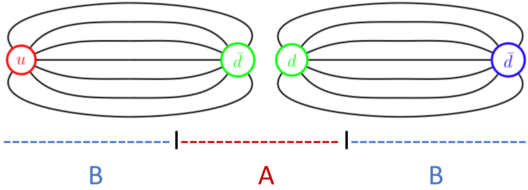
- elementary particle collision experiments such as $e^+ \; e^-$ collisions show some thermal-like features
- particle multiplicities well described by thermal model




[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by collisions unlikely
- more thermal-like features difficult to understand in PYTHIA
  [Fischer, Sjöstrand (2017)]
- alternative explanations needed

# $QCD\ strings$




- particle production from QCD strings
- Lund string model (e. g. PYTHIA)
- different regions in a string are entangled
- subinterval $A$ is described by reduced density matrix

$$\rho_A = \mathrm{Tr}_B\{\rho\}$$

- reduced density matrix is of mixed state form
- could this lead to thermal-like effects?

# *Microscopic model*

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2}\mathrm{tr}\,\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}$$

- fermionic fields $\psi_i$ with sums over flavor species $i = 1, \ldots, N_f$
- SU($N_c$) gauge fields $\mathbf{A}_\mu$ with field strength tensor $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling $g$ has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for $N_c \to \infty$ with $g^2 N_c$ fixed
  ['t Hooft (1974)]

# *Schwinger model*

- QED in 1+1 dimension

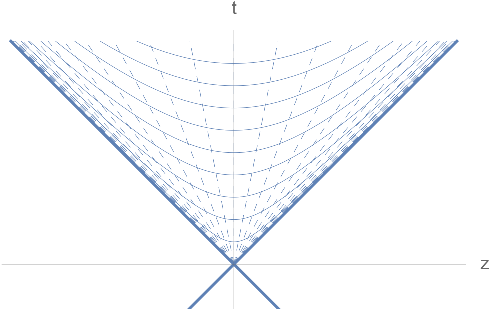
$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly
  [Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \Big\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 \\ - \frac{m\, q\, e^\gamma}{2\pi^{3/2}} \cos\left(2\sqrt{\pi}\phi + \theta\right) \Big\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- scalar mass related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model $m = 0$ leads to free bosonic theory

# *Expanding string solution 1*




- external quark-anti-quark pair on trajectories $z = \pm t$
- coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- symmetry with respect to longitudinal boosts $\eta \to \eta + \Delta\eta$

# *Expanding string solution 2*

- Schwinger boson field depends only on $\tau$

$$\bar{\phi} = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau} \partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field $E = q\phi/\sqrt{\pi}$ must approach the U(1) charge of the external quarks $E \to q_\mathsf{e}$ for $\tau \to 0_+$

$$\bar{\phi}(\tau) \to \frac{\sqrt{\pi} q_\mathsf{e}}{q} \qquad (\tau \to 0_+)$$

- solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \frac{\sqrt{\pi} q_\mathsf{e}}{q} J_0(M\tau)$$

# *Gaussian states*

- theories with quadratic action often have Gaussian density matrix
- fully characterized by field expectation values

$$\bar{\phi}(x) = \langle \phi(x) \rangle, \qquad \bar{\pi}(x) = \langle \pi(x) \rangle$$

  and connected two-point correlation functions, e. g.

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if $\rho$ is Gaussian, also reduced density matrix $\rho_A$ is Gaussian

# *Entanglement entropy for Gaussian state*

- entanglement entropy of Gaussian state in region $A$
  [Berges, Floerchinger, Venugopalan, JHEP 1804 (2018) 145]

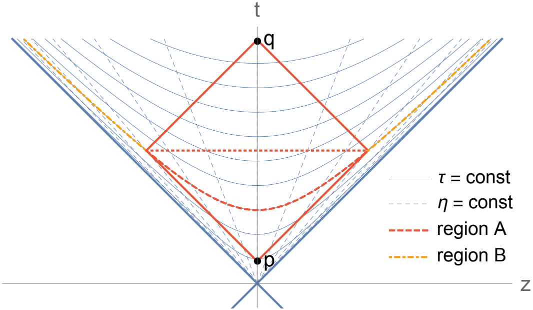
$$S_A = \frac{1}{2}\mathsf{Tr}_A \left\{ D \ln(D^2) \right\}$$

- operator trace over region $A$ only
- matrix of correlation functions

$$D(x,y) = \begin{pmatrix} -i\langle\phi(x)\pi(y)\rangle_c & i\langle\phi(x)\phi(y)\rangle_c \\ -i\langle\pi(x)\pi(y)\rangle_c & i\langle\pi(x)\phi(y)\rangle_c \end{pmatrix}$$

- involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- expectation value $\bar{\phi}$ does not appear explicitly
- coherent states and vacuum have equal entanglement entropy $S_A$

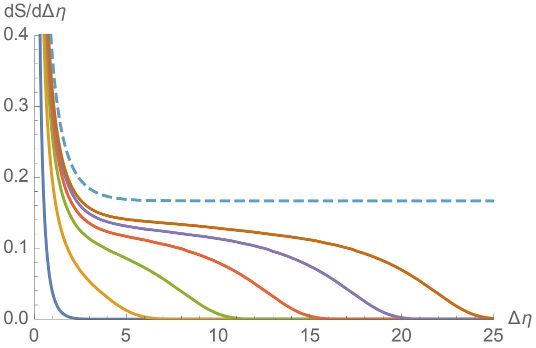
# *Rapidity interval*




- consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time $\tau$
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

# *Bosonized massless Schwinger model*

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{q}{\sqrt{\pi}}$)




$M\tau = 1$, $10^{-1}$, $10^{-2}$, $10^{-3}$, $10^{-4}$, and $10^{-5}$

## *Conformal limit*

- For $M\tau \to 0$ one has conformal field theory limit
  [Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln\left(\Delta z/\epsilon\right) + \text{constant}$$

  with small length $\epsilon$ acting as UV cutoff.

- Here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln\left(2\tau \,\mathsf{sinh}(\Delta\eta/2)/\epsilon\right) + \text{constant}$$

- Conformal charge $c = 1$ for free massless scalars or Dirac fermions.
- Additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) =& \frac{c}{6} \mathsf{coth}(\Delta\eta/2) \\ \to& \frac{c}{6} \qquad\qquad (\Delta\eta \gg 1) \end{aligned}$$

- Entropy becomes extensive in $\Delta\eta$ !

# *Universal entanglement entropy density*

- for very early times "Hubble" expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

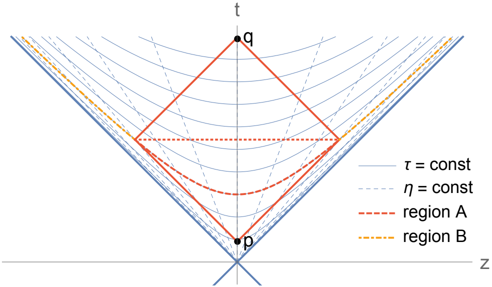
  with conformal charge $c$
- for QCD in 1+1 D (gluons not dynamical, no transverse excitations)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

# *Modular or entanglement Hamiltonian 1*




- conformal field theory
- hypersurface $\Sigma$ with boundary on the intersection of two light cones
- reduced density matrix [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \qquad Z_A = \mathrm{Tr}\, e^{-K}$$

- modular or entanglement Hamiltonian $K$

# *Modular or entanglement Hamiltonian 2*

- modular or entanglement Hamiltonian is **local expression**

$$K = \int_\Sigma d\Sigma_\mu \, \xi_\nu(x) \, T^{\mu\nu}(x).$$

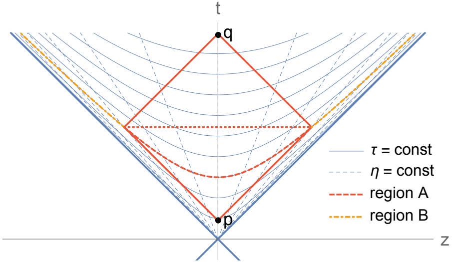
- energy-momentum tensor $T^{\mu\nu}(x)$ of excitations
- vector field

$$\xi^\mu(x) = \tfrac{2\pi}{(q-p)^2} [(q-x)^\mu (x-p)(q-p) \\ + (x-p)^\mu (q-x)(q-p) - (q-p)^\mu (x-p)(q-x)]$$

  end point of future light cone $q$, starting point of past light cone $p$
- inverse temperature and fluid velocity

$$\xi^\mu(x) = \beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

# *Modular or entanglement Hamiltonian 3*




- for $\Delta\eta \to \infty$: fluid velocity in $\tau$-direction, $\tau$-dependent temperature

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

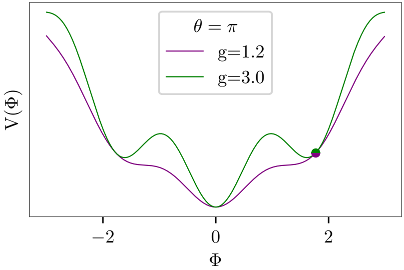
- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \hbar c/(2\pi x)$

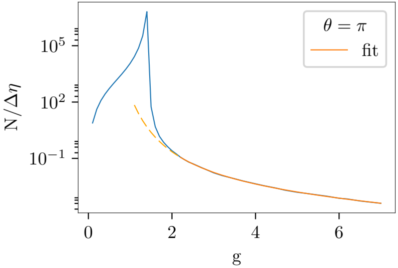
# *Physics picture*

- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically limits $\Delta\eta \to \infty$ and $M\tau \to 0$ do not commute
  - $\Delta\eta \to \infty$ for any finite $M\tau$ gives pure state
  - $M\tau \to 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

# *Particle production in massive Schwinger model*

[ongoing work with Lara Kuhn, Jürgen Berges]




- for expanding strings
- asymptotic particle number depends on $g \sim m/q$
- exponential suppression for large fermion mass $g \gg 1$

$$\frac{N}{\Delta\eta} \sim e^{-0.55\frac{m}{q}+7.48\frac{q}{m}+\ldots} = e^{-0.55\frac{m}{\sqrt{2\sigma}}+7.48\frac{\sqrt{2\sigma}}{m}+\ldots}$$

# *Wigner distribution and entanglement*

- Classical field approximation usually based on non-negative Wigner representation of density matrix
- leads for many observables to classical statistical description
- can nevertheless show entanglement and pass Bell test for "improper" variables where Weyl transform of operator has values outside of its spectrum [Revzen, Mello, Mann, Johansen (2005)]
- Bell test violation also possible for negative Wigner distribution [Bell (1986)]

# *Transverse coordinates*

- so far dynamics strictly confined to 1+1 dimensions
- transverse coordinates may fluctuate, can be described by Nambu-Goto action ($h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$)

$$
\begin{aligned}
S_{\mathsf{NG}} &= \int d^2x \sqrt{-\mathsf{det}\, h_{\mu\nu}}\, \{-\sigma + \dots\} \\
&\approx \int d^2x \sqrt{g} \left\{ -\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \dots \right\}
\end{aligned}
$$

- two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates $X^i$ with $i = 1, 2$

# *Temperature and entanglement entropy*

- for conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- for static interval of length $L$ [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T,l) = \frac{c}{3} \ln \left( \frac{1}{\pi T \epsilon} \sinh(\pi L T) \right) + \text{const}$$

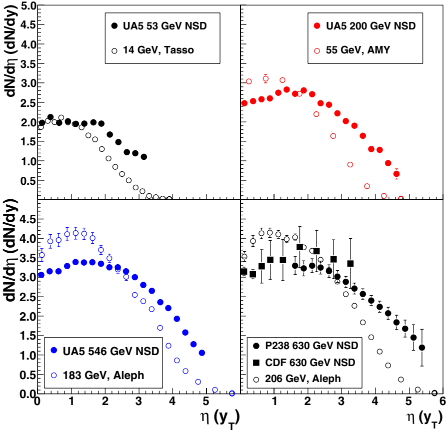
- compare this to our result in expanding geometry

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left( \frac{2\tau}{\epsilon} \,\mathsf{sinh}(\Delta\eta/2) \right) + \text{const}$$

- expressions agree for $L = \tau \Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

# *Rapidity distribution*




[open (filled) symbols: $e^+e^-$ (pp), Grosse-Oetringhaus & Reygers (2010)]

- rapidity distribution $dN/d\eta$ has plateau around midrapidity
- only logarithmic dependence on collision energy

# *Experimental access to entanglement ?*

- could longitudinal entanglement be tested experimentally?
- unfortunately entropy density $dS/d\eta$ not straight-forward to access
- measured in $e^+e^-$ is the number of charged particles per unit rapidity $dN_{\mathsf{ch}}/d\eta$ (rapidity defined with respect to the thrust axis)
- typical values for collision energies $\sqrt{s} = 14 - 206$ GeV in the range

$$dN_{\mathsf{ch}}/d\eta \approx 2 - 4$$

- entropy per particle $S/N$ can be estimated for a hadron resonance gas in thermal equilibrium $S/N_{\mathsf{ch}} = 7.2$ would give

$$dS/d\eta \approx 14 - 28$$

- this is an upper bound: correlations beyond one-particle functions would lead to reduced entropy

# *Entanglement and QCD physics*

- how strongly entangled is the nuclear wave function?
- what is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]
- does saturation at small Bjorken-$x$ have an entropic meaning?
- entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015); Kovner, Lublinsky, Serino (2018)]
- could entanglement entropy help for a non-perturbative extension of the parton model?
- entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]