Perfect-fluid hydrodynamics of particles with spin 1/2 (with classical treatment of spin)

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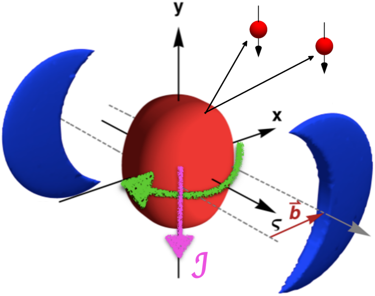
more in: W.F., A. Kumar, R.Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

"Theoretical Foundations of Relativistic Hydrodynamics"
Banff, Nov. 24-29, 2019, Canada

Spin polarization of Lambdas in heavy-ion collisions (I)

- Noncentral HIC create fireballs with large global angular momenta $\sim 10^5\hbar$
  F. Becattini, F. Piccinini, J. Rizzo, PRC 77 (2008) 024906, [0711.1253]

- Vorticity of the system may produce a spin polarization of the produced particles
  S. J. Barnett, Rev. Mod. Phys. 7, 129 (1935)
  A. Einstein and W. de Haas, Deutsche Physikalische Gesellschaft, Verhandlungen 17, 152 (1915)



Spin polarization of Lambdas in heavy-ion collisions (II)

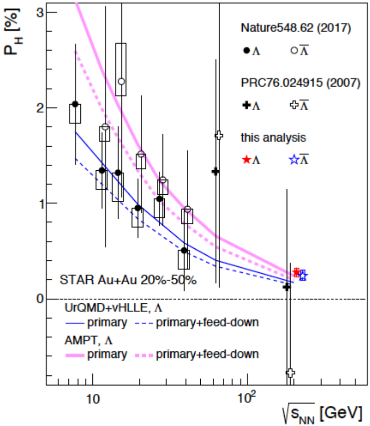
- The first positive measurement of global spin polarization of $\Lambda/\bar{\Lambda}$ hyperons
  STAR , Nature 548 (2017) 62-65, [1701.06657]

spin polarization
$\updownarrow$
vorticity

- Statistical formalism describes the global polarization
  D. N. Zubarev, A. V. Prozorkevich, S. A. Smolyanskii, Theoret. and Math. Phys. 40 (1979), 821
  F. Becattini, L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75 (2015) 191 [1403.6265]

spin polarization tensor $\omega^{\mu\nu}$
$\Updownarrow$ global equilibrium, with asymmetric $T^{\mu\nu}$
thermal vorticity $\varpi^{\mu\nu}$

$$\varpi_{\mu\nu} = -\tfrac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu),$$
$$\beta^\mu = u^\mu / T$$



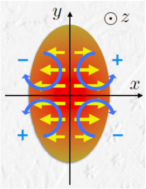
STAR, PRC 98 (2018) 014910, [1805.04400]
UrQMD+vHLLE: I. Karpenko, F. Becattini, EPJC 77, 213 (2017), [1610.04717]
AMPT: H. Li, L. Pang, Q. Wang, and X. Xia, PRC 96, 054908 (2017), [704.01507]

spin polarization tensor $\omega^{\mu\nu}$ = spin potential $\Omega^{\mu\nu}$ divided by temperature T, $\omega^{\mu\nu} = \Omega^{\mu\nu}/T$

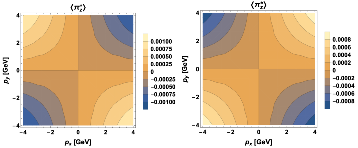
Spin polarization of Lambdas in heavy-ion collisions (III)

- The quadrupole structure of longitudinal polarization is not described within current approach

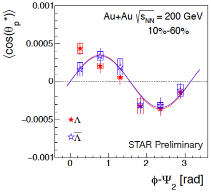
- Some solution of the sign problem found within thermal models
  S. Voloshin, EPJ Web Conf. 17 (2018) 10700, [1710.08934]
  W. F., A. Kumar, R. Ryblewski, A. Mazeliauskas, [1904.00002]

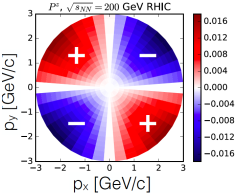


S. Voloshin, SQM2017



(a) (b)





Let us incorporate spin into a hydrodynamic framework!

- Present works limited to the calculation of polarization at freeze-out

  $\rightarrow$ space-time dynamics of spin polarization?

- HIC evolution best described within relativistic hydrodynamics

  $\rightarrow$ inclusion of polarization effects in hydrodynamics?

local thermodynamic equilibrium
where $\omega_{\mu\nu} = \varpi_{\mu\nu}$
replaced/extended to
local spin-thermodynamic equilibrium
where the dynamics of $\omega_{\mu\nu}$ is governed by the conservation law for angular momentum

spin polarization tensor becomes equal to the thermal vorticity if dissipation included (???)
Torrieri,...

First steps have been already made...

- Relativistic hydrodynamics with spin formulated recently
  W. F., B. Friman, A. Jaiswal, E. Speranza, PRC 97 (4) (2018) 041901. [1705.00587]
  W. F., B. Friman, A. Jaiswal, R. Ryblewski, E. Speranza, PRD 97 (11) (2018) 116017. [1712.07676]

  follow up study: K. Hattori, M. Hongo, X-G. Huang, M. Matsuo, H.Taya [1901.06615]

  → particular choice of the forms of energy-momentum $T^{\mu\nu}$ and spin tensors $S^{\lambda\mu\nu}$

  → recent works clarified the use of de Groot - van Leeuwen - van Weert (GLW) expressions
  W. F., A. Kumar, R. R., PRC 98 (2018) 044906. [1806.02616]
  F. Becattini, W. F., E. Speranza, PLB 789, (2019) 419-425 [1807.10994],

- Realistic applications of the GLW-based hydrodynamics with spin to HIC in the small-polarization limit is in progress
  W. F., A. Kumar, R. R. and R. Singh, PRC 99 (2019) 044910
  W. F., A. Kumar, R. R. and R. Singh, forthcoming

Spin polarization tensor $\omega_{\mu\nu}$ — physics interpretation

- Becattini and followers: mean spin polarization of particles with momentum $p^\mu = (E_p, \boldsymbol{p})$ at the space-time point $x^\mu = (t, \boldsymbol{x})$ [defined in the particle rest frame, PRF, denoted below by an asterisk *]

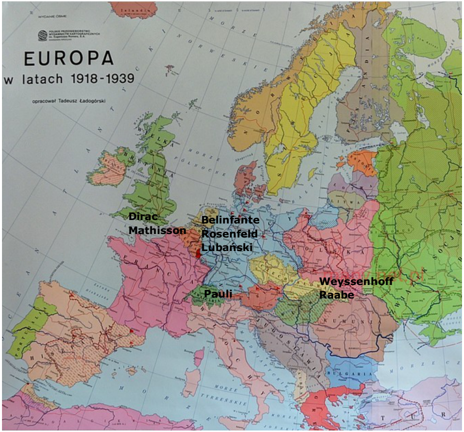
$$\langle \boldsymbol{P}(x,p) \rangle = -\frac{1}{4}\boldsymbol{P}$$

$$\boldsymbol{P} = \frac{1}{m}\left[E_p\, \boldsymbol{b} - \boldsymbol{p} \times \boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p + m}\boldsymbol{p}\right] = \boldsymbol{b}_*$$

- magnetic-like component in PRF

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}.$$

The Weyssenhoff circle, years 1930s - 1940s



Jan Weyssenhoff
1889-1972

The Weyssenhoff hydrodynamic model (I)

J. Weyssenhoff and A. Raabe, Acta Phys. Pol. 9 (1947) 7

Vol. IX (1947) Acta Physica Polonica Fasc. 1

RELATIVISTIC DYNAMICS OF SPIN-FLUIDS AND SPIN-PARTICLES*

By Jan WEYSSENHOFF and A. RAABE †, Institute of Theoretical Physics, Jagellonian University, Kraków **

1. Following up a train of thought inaugurated by Einstein and Gromer, Mathisson *(1, 1937)* and Lubański *(2, 1937)* deduced the equation of motion of a material particle endowed with spin from the general principles of the theory of relativity. Even for a free particle in Galileian domains these equations do not coincide with the Newtonian laws of motion; there remains an additional term depending on the internal angular momentum or spin of the particle, which raises the order of these differential equations to three. Mathisson did not notice the fact that for the above conditions his equations are equivalent to the equations which were previously found by Frenkel *(3, 1926)* for a spinning electron (if only their terms depending on the electromagnetic field are dropped). This is all the more excusable as Frenkel considered throughout the additional term in his equations as an infinitely small perturbation, and did not even mention the fact that they disagree with Newton's First Law for a free particle.

In the present paper we give a third method of obtaining the same equations by establishing first the laws of the dynamics of an incoherent spin-fluid and passing then to the limit. Strictly speaking, we obtain different and much simpler equations, which prove however to be equivalent to Frenkel's and Mathisson's. The simplification is due to the explicit introduction of the 4-vector of linear monumentum and energy G^α.

* Presented at a meeting of the Cracow Section of the Polish Physical Society on February 28, 1945. The main contents of this paper and the next one, as well as most of the results of the three following, were a subject of a lecture at a secret meeting of physicists at Prof. Pieńkowski's home in Warsaw, October 1942.

** Mr. Raabe was a highly gifted young physicist with whom I outlined in all its main features the contents of this paper and the next one in 1940/41 in Lwów. We tried to pursue our work in 1942 in Cracow, but unfortunately in June 1942 Mr. Raabe fell victim of a man-hunt in the streets of Cracow; he died four months later in the German concentration camp of Oświęcim. (J. W.)

The Weyssenhoff hydrodynamic model (II)

J. Weyssenhoff and A. Raabe, Acta Phys. Pol. 9 (1947) 7

1) conservation of energy and momentum with an asymmetric energy-momentum tensor

$$T^{\mu\nu}(x) = g^{\mu}(x)u^{\nu}(x), \quad \partial_\nu T^{\mu\nu}(x) = 0$$

u^μ is the four-velocity of the fluid element
g^μ is the density of four-momentum
with the notation $\partial_\nu(fu^\nu) \equiv Df$ we may write $Dg^\mu = 0$

2) conservation of total angular momentum $J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$ (orbital and spin parts)

$$L^{\lambda,\mu\nu}(x) = x^{\mu}T^{\nu\lambda}(x) - x^{\nu}T^{\mu\lambda}(x), \quad S^{\lambda,\mu\nu}(x) = s^{\mu\nu}(x)u^{\lambda}(x)$$

$s^{\mu\nu} = -s^{\nu\mu}$ describes the spin density

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \rightarrow Ds^{\mu\nu} = g^\mu u^\nu - g^\nu u^\mu$$

3) 10 equations for 13 unknown functions: g^μ, $s^{\mu\nu}$ and u^i $(i = 1, 2, 3)$
additional constraint has been adopted, the Frenkel (or Weyssenhoff) condition $s^{\mu\nu}u_\mu = 0$
this gives g^μ in the form

$$g^\mu = \mu u^\mu - s^{\mu\nu}a_\nu, \quad D(\mu u^\mu - s^{\mu\nu}a_\nu) = 0,$$

where a_ν is the four-acceleration

Origin of this all: internal angular momentum tensor

Internal angular momentum tensor of a particle
M. Mathisson, APPB 6 (1937) 163–2900

$$\boxed{s^{\alpha\beta} = \stackrel{\textit{fluid element}}{\longrightarrow} s^{\alpha\beta}(x)}$$

The **Frenkel (or Weyssenhoff) condition**

$$\boxed{s^{\alpha\beta} p_\alpha = 0 \quad \stackrel{\textit{fluid element}}{\longrightarrow} \quad s^{\alpha\beta}(x) u_\alpha(x) = 0}$$

Spin vector, since $s \cdot p = 0$ one can introduce the spin four-vector

$$s^\alpha = \frac{1}{2m} \epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta} \qquad \stackrel{PRF\;\left[p^\mu=(m,0,0,0)\right]}{\Longrightarrow} \qquad s^\alpha = (0, \boldsymbol{s}_*)$$

Quantum-mechanics input: for spin-1/2 particles the length of $\boldsymbol{s}_*$ is

$$\boxed{|\boldsymbol{s}_*|^2 = \cancel{\mathfrak{s}}^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}}$$

Changing gears, modern (?) kinetic-theory perspective, angular momentum conservation

We identify collisional invariants of the Boltzmann equation (BE)

In addition to four-momentum and conserved charges one can include the **total angular momentum**

$$j_{\alpha\beta} = l_{\alpha\beta} + s_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha + s_{\alpha\beta}$$

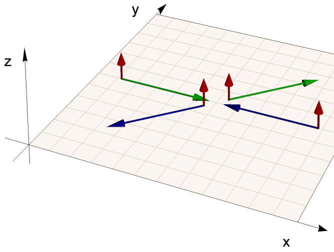
The **locality of the standard BE** suggests that the **orbital part can be eliminated**, and **the spin part can be considered separately**.

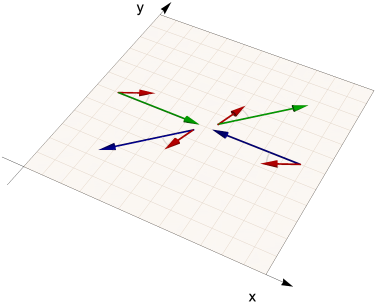
For elastic binary collisions of particles 1 and 2 going to $1'$ and $2'$, this suggests that
C. G. van Weert, Henkes- Holland N.V. – Haarlem, 1970.

$$\boxed{s_1^{\alpha\beta} + s_2^{\alpha\beta} = s_{1'}^{\alpha\beta} + s_{2'}^{\alpha\beta}}$$

Spin collisions

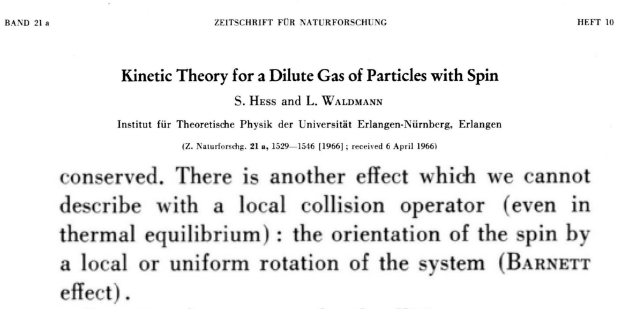
Conservations of internal angular momentum admits two types of simple solutions: either the **sum of two spin three-vectors or their difference (before and after the collision) vanishes**. They may be interpreted as collisions in the spin **singlet** and **triplet** states.





Non-local effects?

Locality of the collision kernel suggests that we cannot describe the Barnett effect - the fact realized in 1966.



BAND 21 a — ZEITSCHRIFT FÜR NATURFORSCHUNG — HEFT 10

Kinetic Theory for a Dilute Gas of Particles with Spin

S. Hess and L. Waldmann

Institut für Theoretische Physik der Universität Erlangen-Nürnberg, Erlangen

(Z. Naturforschg. **21 a**, 1529—1546 [1966]; received 6 April 1966)

conserved. There is another effect which we cannot describe with a local collision operator (even in thermal equilibrium): the orientation of the spin by a local or uniform rotation of the system (Barnett effect).

Further developments require the use of non-local collisional kernels.

A. Jaiswal, R. S. Bhalerao, S. Pal, PLB 720 (2013) 347–351

for massless particles, chiral kinetic theory, M. Stephanov et al.

Spin-dependent distribution functions and spin invariant measure

We introduce a **spin-dependent equilibrium distribution functions for particles and antiparticles**

$$f^{\pm}_{\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p\cdot\beta(x) + \frac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right)$$

with $\beta^\mu = u^\mu/T$ and $\xi^\mu = \mu/T$.

Different orientations of spin can be integrated out with the help of a **covariant measure**

$$\int dS \ldots = \frac{m}{\pi\,\mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\,\delta(p\cdot s)\ldots$$

The prefactor $m/(\pi\,\mathfrak{s})$ is chosen to obtain the **normalization**

$$\int dS = \frac{m}{\pi\,\mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\,\delta(p\cdot s) = 2$$

Charge current

The **charge current** is obtained from the generalization of the standard definition

$$N^\mu_{\rm eq} = \int dP \int dS\, p^\mu \left[f^+_{\rm eq}(x,p,s) - f^-_{\rm eq}(x,p,s)\right]$$

which after using the forms of the equilibrium functions leads to

$$\boxed{N^\mu_{\rm eq} = 2\sinh(\xi) \int dP\, p^\mu e^{-p\cdot\beta} \int dS\, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)}$$

For **small values of** ω one gets

$$N^\mu_{\rm eq} \quad = \quad 2\sinh(\xi) \int dP\, p^\mu\, e^{-p\cdot\beta} \int dS \left(1 + \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) \quad = \quad 2\left(e^{\mu/T} - e^{-\mu/T}\right) \int dP\, p^\mu\, e^{-p\cdot\beta}$$

spin effects start with quadratic terms in ω
the leading term in ω **gives the standard expression for the charge current**

Energy-momentum tensor

The **energy-momentum tensor** is given by

$$T^{\mu\nu}_{\rm eq} = \int dP \int dS \; p^\mu p^\nu \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s) \right]$$

which leads to

$$\boxed{T^{\mu\nu}_{\rm eq} = 2\cosh(\xi) \int dP \, p^\mu p^\nu \, e^{-p\cdot\beta} \int dS \; \exp\left(\frac{1}{2}\omega_{\alpha\beta} s^{\alpha\beta}\right)}$$

For **small values of** ω one gets

$$T^{\mu\nu}_{\rm eq} \quad = \quad 2\cosh(\xi) \int dP \, p^\mu p^\nu \, e^{-p\cdot\beta} \quad = \quad 4\cosh(\xi) \int dP \, p^\mu p^\nu \, e^{-p\cdot\beta}$$

again, spin effects start with quadratic terms in ω
the leading term in ω gives the standard expression for the charge current

N^μ and $T^{\mu\nu}$ for arbitrarily large polarization

Let us consider the integral for **arbitrary values of** ω

$$\int dS \, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

Using definition of the dual polarization tensor $\tilde{\omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta}$ one may write

$$\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta} = \frac{p_\gamma \tilde{\omega}^{\gamma\delta}}{m}\, s_\delta \quad \overset{PRF}{\Rightarrow} \quad \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta} = \boldsymbol{b}_* \cdot \boldsymbol{s}_* = \boldsymbol{P} \cdot \boldsymbol{s}_*$$

For **arbitrary values of** ω one gets

$$\int dS \, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = \int_{-1}^{+1} e^{\mathfrak{s} P x} dx = \frac{2\sinh(\mathfrak{s}P)}{\mathfrak{s}\,P} \qquad \text{with} \qquad P = |\boldsymbol{P}| = |\boldsymbol{b}_*|$$

Since P depends on momentum, $T^{\mu\nu}$ has no longer a perfect-fluid form

Large values of ω induce momentum anisotropy $\rightarrow$ anisotropic hydrodynamics with spin

W. F., R. Ryblewski, PRC 83 (2011) 034907. [1007.0130]

M. Martinez, M. Strickland, NPA 848 (2010) 183–197. [1007.0889]

Spin tensor

The **spin tensor** is defined as an expectation value of the internal angular momentum tensor,

$$S^{\lambda\mu\nu}_{\rm eq} = \int dP \int dS \; p^\lambda \, s^{\mu\nu} \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s) \right]$$

$$\boxed{S^{\lambda\mu\nu}_{\rm eq} = 2\cosh(\xi) \int dP \, p^\lambda \, e^{-p\cdot\beta} \int dS \, s^{\mu\nu} \, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)}$$

For **small values of** ω one gets

$$\int dS \, s^{\mu\nu} \left(1 + \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = \frac{\omega_{\alpha\beta}}{2m^2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}p_\rho p_\gamma \int dS \, s_\delta s_\sigma = \frac{2}{3m^2}\,\mathfrak{s}^2\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right)$$

$$S^{\lambda\mu\nu}_{\rm eq} \quad = \quad \frac{4\,\mathfrak{s}^2}{3m^2}\cosh(\xi)\int dP \, p^\lambda \, e^{-p\cdot\beta}\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right)$$

the leading term in ω reproduces $S^{\lambda,\mu\nu}_{\rm eq}$ obtained earlier from the quantum kinetic equation in the GLW version (connected with the canonical one by a pseudo-gauge transformation)

$S^{\lambda\mu\nu}$ - case of arbitrarily large polarization

Let us consider the integral for arbitrary value of ω

$$\int dS\, s^{\mu\nu} \, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

In this case one can find

$$\int dS\, s^{\mu\nu} \, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = \frac{\chi(P\mathfrak{s})}{m^2}\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \alpha}\right)$$

with

$$\chi(P\mathfrak{s}) = \mathfrak{s}^2\,\frac{2\sinh(P\mathfrak{s})}{P\mathfrak{s}}\,\frac{L(P\mathfrak{s})}{P\mathfrak{s}} \quad \text{where} \quad L(x) = \coth(x) - \frac{1}{x} \quad \text{is the Langevin function}$$

For **arbitrary values of** ω one gets

$$S^{\lambda,\mu\nu}_{\rm eq} \quad = \quad \frac{2}{m^2}\cosh(\xi)\int dP\, p^\lambda\, e^{-p\cdot\beta}\chi(P\mathfrak{s})\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \alpha}\right)$$

Pauli-Lubański vector (I)

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J. K. LUBAŃSKI

Obituary notice by L. ROSENFELD, Manchester.

The circle of Lubański's friends was not large. During the wartime which was so difficult for him he lived very retired and he was rather shy and taciturn. But those who were in closer contact with him have been able to discover his sensitive and refined personality. After the liberation a striking change came over him, that made us only then understand how much he had suffered during the war. He surprised us by an unknown alacrity and optimism. His new task in Delft gave him great satisfaction and he fulfilled it with an enthusiasm and energy that authorised the greatest expectations for his scientific and personal future. This made the shock the more violent for his friends when the news reached them of his unexpected death on the 8-th December 1946, after only a very short illness.

Jozeph Kazimir Lubański was born in 1914 in Rumania from Polish parents. He spent his youth in Russia; only in 1926 did he come to Poland where he studied Physics at the Universities of Wilno and Kraków. In Kraków he worked under the direction of the very original theoretician Mathisson (who died in England during the war); his first paper in 1937 is based on Mathisson's theories. Until the autumn of 1938 Lubański was assistant at the Institute of Theoretical Physics at Wilno. In December 1938 he came to Leiden with a stipend from the Polish Government to work under the direction of Professor Kramers. Since that time he lived in Holland, where he was greatly helped during the war by the Lorentz Fund. As a Polish citizen he was forced already in 1940 to leave the coastal region and after a short stay in the country-side he settled in Utrecht, where he stayed until his appointment at Delft in October 1945.

In Leiden he worked with Kramers and Belinfante on the Theory of Particles with arbitrary spin. His investigations on this subject are set-out in three papers, published in the Dutch journal Physica. These papers witness his profound knowledge of the abstract theories of modern algebra and his mastery in applying them to fundamental problems of theoretical physics. The same qualities characterize his further publications which contained the results of the work he did in Utrecht.

Pauli-Lubański vector (II)

We introduce **phase-space density of the Pauli-Lubański (PL) vector**

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32–49. [1303.3431]

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\left(\Delta\Sigma_\lambda(x)\, E_p \frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p}\right)\frac{p^\beta}{m}$$

where

$$E_p \frac{d\Delta\mathcal{S}^{\mu\nu}}{d^3p} = \frac{\cosh(\xi)}{4\pi^3}\Delta\Sigma\cdot p\, e^{-p\cdot\beta}\int dS\, s^{\mu\nu}\,\exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

We define the **particle number current** for both particles and antiparticles as

$$\mathcal{N}^\mu_{\rm eq} = \int dP \int dS\; p^\mu\left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right].$$

Using this expression we obtain the momentum density of the total number of particles

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\cosh(\xi)}{4\pi^3}\Delta\Sigma\cdot p\, e^{-p\cdot\beta}\int dS\,\exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

The **mean PL vector** is obtained as the ratio

$$\boxed{\pi_\mu(x,p) = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{\int dS\, s^{\nu\alpha}\,\exp\left(\frac{1}{2}\omega_{\rho\sigma}s^{\rho\sigma}\right)}{\int dS\,\exp\left(\frac{1}{2}\omega_{\rho\sigma}s^{\rho\sigma}\right)}\frac{p^\beta}{m}}$$

Pauli-Lubański vector (III)

For **arbitrary values of the polarization** one gets

$$\pi_\mu = -\mathfrak{s}^2 \frac{\tilde{\omega}_{\mu\beta} p^\beta}{m} \frac{L(P\,\mathfrak{s})}{P\,\mathfrak{s}}$$

In PRF

$$\pi_*^0 = 0, \quad \boldsymbol{\pi}_* = -\mathfrak{s}^2 \boldsymbol{P} \frac{L(P\,\mathfrak{s})}{P\,\mathfrak{s}}$$

For small and large P Langevin function is having the form

$$L \approx 1 \quad \text{for} \quad x \gg 1 \quad \text{and} \quad L \approx \frac{x}{3} \quad \text{for} \quad x \ll 1$$

respectively, thus we obtain two important results:

$$\boldsymbol{\pi}_* = -\mathfrak{s} \frac{\boldsymbol{P}}{P}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s} = \sqrt{\frac{3}{4}}, \quad \text{if} \quad P \gg 1$$

The normalization of the PL vector cannot exceed the value of $\mathfrak{s}$.

$$\boldsymbol{\pi}_* = -\mathfrak{s}^2 \frac{\boldsymbol{P}}{3}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s}^2 \frac{P}{3} = \frac{P}{4}, \quad \text{if} \quad P \ll 1$$

For small values of P the classical treatment of spin reproduces the quantum mechanical result

Entropy conservation á la mode de Boltzmann

Classical treatment of spin allows for explicit derivation of the **entropy current conservation**. We adopt the Boltzmann definition

$$\boxed{H^\mu = -\int dP \int dS\, p^\mu \left[f^+_{\rm eq} \left(\ln f^+_{\rm eq} - 1 \right) + f^-_{\rm eq} \left(\ln f^-_{\rm eq} - 1 \right) \right]}$$

Using form of $f^\pm$ and the conservation laws for energy, linear and angular momentum, and charge, we obtain

$$H^\mu = \beta_\alpha T^{\mu\alpha}_{\rm eq} - \frac{1}{2} \omega_{\alpha\beta} S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu_{\rm eq}$$

$$\partial_\mu H^\mu = \left(\partial_\mu \beta_\alpha \right) T^{\mu\alpha}_{\rm eq} - \frac{1}{2} \left(\partial_\mu \omega_{\alpha\beta} \right) S^{\mu,\alpha\beta}_{\rm eq} - \left(\partial_\mu \xi \right) N^\mu_{\rm eq} + \partial_\mu \mathcal{N}^\mu_{\rm eq}$$

With the help of the relation $\mathcal{N}^\mu_{\rm eq} = \frac{\cosh(\xi)}{\sinh(\xi)} N^\mu_{\rm eq}$ and the conservation of charge one can easily show that

$$\boxed{\partial_\mu H^\mu = 0}$$

Contributions to H^μ, connected with the polarization tensor, start with **quadratic terms in** ω.

If we restrict ourselves to **linear terms in** ω, all thermodynamic quantities become independent of ω, while the conservation of the angular momentum determines the polarization evolution in a given hydrodynamic background.

Conclusions

- classical treatment of spin (which follows ideas of Mathisson, Weyssenhoff, and Raabe from 1930s and 1940s) combined with basic ideas of classical transport theory yields a consistent hydrodynamic framework for particles with spin 1/2
- the obtained scheme is consistent with Becattini's ideas if: 1) spin polarization tensor is allowed to be independent of thermal vorticity, 2) spin polarization is small, and 3) one adopts the GLW (de Groot - van Leuwen - van Weert) forms of the energy-momentum and spin tensors
- for small spin polarization, our classical treatment is consistent with a quantum description, provided $\mathfrak{s} = \sqrt{3/4}$
- maximum spin polarization is bounded by $\sqrt{3/4}$ rather than by $1/2$ — a consequence of classical description of spin
- one can prove that the Boltzmann entropy is strictly conserved $\rightarrow$ perfect-fluid with spin
- good starting point for inclusion of the dissipation effects such as viscosity of the spin fluids etc.