A Riemann-Roch theorem in Bott-Chern cohomology

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Banff, October 28th - November 1st 2019

Bridging the gap between Kähler and non-Kähler complex geometry

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- In general $H_{\mathrm{BC}}^{\cdot}(X,\mathbf{C})$ strictly finer than $H_{\mathrm{DR}}^{\cdot}(X,\mathbf{C})$.
- $H_{\mathrm{BC}}^{(=)}(S,\mathbf{R}) = \bigoplus_{0\le p\le n} H_{\mathrm{BC}}^{(p,p)}(S,\mathbf{R})$.

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- It will be denoted $\mathrm{ch}_{\mathrm{BC}}(E) \in H^{(=)}_{\mathrm{BC}}(S, \mathbf{R})$.

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- If $R^{\cdot}p_*F$ locally free,

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- For $c_{1,\mathrm{BC}}\left(R^{\cdot}p_*F\right)$, the result is valid in full generality.

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- If M, S projective, the result follows from versions of Riemann-Roch-Grothendieck, even if $R^{\cdot}p_*F$ not locally free.
- In general, if $\mathscr{F}$ coherent sheaf, $\mathrm{ch}_{\mathrm{BC}}(\mathscr{F})$ was in principle defined by Schweitzer.

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- …so as to obtain a nondegenerate Hermitian form of signature (∞, ∞).

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- η is a Hermitian form of signature (∞, ∞).

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- $\ldots$ which is nilpotent.

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- $[d, d^*] = 0, \left[\overline{\partial}, \overline{\partial}^*\right] = -i\overline{\partial}\partial\omega$.
- Holomorphic Laplacian vanishes if and only if $\overline{\partial}\partial\omega = 0$.

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- In families, smooth deformation destroys the holomorphic structure: Bott-Chern information is lost!

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- By enlarging the set of permissible metrics.

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- This holds in particular when X is Kähler.

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- Exotic Laplacian $\overline{\partial}^X \partial^X \omega^X$ obstruction to local index theorem.

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If $\overline{\partial}^X \partial^X \omega^X = 0$, there is a local index theorem, compatible with RRH.

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- The cohomology of this new complex is still equal to $H^{(0,\cdot)}(X, F)$.

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- $A_b' = \partial^X + \overline{\partial}^{\widehat{TX}*} + i_{\overline{y}}/b^2 + \overline{y}^* \wedge /b^2 - i\partial\omega^X + \ldots$.

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- It is potentially good in families: it has been obtained by replacing L_2 metric by nonpositive metric.

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- Analytic torsion forms $\frac{\overline{\partial}^S \partial^S}{2i\pi} T = \alpha_\infty - \alpha_0 = \mathrm{ch}\left(R^\cdot p_* F, g^{R^\cdot p_* F}\right) - p_* \left[\mathrm{Td}\left(TX, g^{TX}\right)\mathrm{ch}\left(F, g^F\right)\right]$.

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- These results extend to $\overline{\partial}^M \partial^M \omega^M = 0$.

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- This is possible because we deform nondegenerate Hermitian forms.

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- Make $t \to 0\ldots$ finally!

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- ...and explain the given proof in the general case.

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- The term $\overline{\partial}^S\partial^S\omega^S$ appears 'because' it is a Laplacian in the exotic Hodge theory of S.

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- The family of fiberwise Hodge Laplacians is 0 acting on $\mathbf{C}$!

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