Dimers, Ising models and their interactions

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1 Overview of the Field

Problems arising from statistical physics constituted one of the core research areas in probability theory over the past few decades. It was predicted by physicists a long time ago that in two dimensions, such models possess a particularly rich structure with large scale conformal symmetries. In recent years we have witnessed the resolution of several longstanding conjectures in this area, perhaps the most important ones being a proof of convergence of percolation and Ising interfaces to Schramm's SLE curves by Smirnov and that of Loop-erased Random Walks and Uniform Spanning Trees by Lawler, Schramm and Werner. Frequently, the new probabilistic tools seem to provide a completely fresh approach to even known theorems in this area, which shed new light to problems previously perceived intractable. It seems that most new ideas in this area stem from combining tools arising in combinatorics, algebra and complex analysis with exciting modern developments in probability theory. Therefore, exchange of ideas by bringing together mathematicians with expertise in these various facets of the subject should be very exciting and productive.

The focus of this workshop was on two particularly popular models: the *dimer model*, which is a model of perfect matching on graphs and *lattice spin models*, with a particular focus on the *Ising model* and the *random cluster model*.

2 Recent Developments and Open Problems

The dimer model has been a popular one particularly because of the *integrable* nature of the model for planar graph, which beautifully combines together analysis, combinatorics and algebra [4]. In particular, a lot of progress has been possible in the last few decades spearheaded by the work of Kenyon where he analyzed the dimer model fluctuations on lattices exhibiting some strict microscopic symmetry [5]. A plethora of work has been done since then analysing the fine details of the model in various setups [6, 2, 1, 7, 3]. A part of the workshop involved talks reporting recent progress on some of these questions, see for example the talks highlighted in Sections 3.4, 3.5, 3.10, 3.14, 3.15 and 3.16. There has been recent interest in establishing universality for the dimer model fluctuations in general planar graphs, and also for certain simple non-planar setting, for example on a Riemann surface (see Section 3.13). However, one big highlight of the workshop was the combination of talks by Ramassamy (Section 3.11), Laslier and Chelkak (Section 3.12) where they outlined a program to attack this problem using certain special embeddings called s-embeddings. This approach promises far-reaching conclusions and sparked a lot of interest and excitement in the community.

The other topic of interest in this workshop was various spin models on lattices, in particular the Ising model. As is known, there are various height models which do not exhibit strong integrability properties like the dimer model; the six vertex model being the most notable example. Michael Aizenman (Section 3.2) talked about quantum spin models and how it is related to something called the *dimerization phenomenon*. Various other talks were on topics related to the six-vertex model (Sections 3.1, 3.3, 3.7, 3.8, 3.9, 3.17, 3.19). Some of these talks expanded on the connection of these models with the Ising model and the more general Ashkin–Teller model.

We now outline some of the open problems which arose from the workshop.

2.1 Benjamin Young

Consider a finite bipartite graph G, with boundary vertices colored cyclically in red, then green, then blue. Consider double-dimer configurations such that the boundary vertices are endpoints of double-dimer paths, and such that the connection of boundary vertices is "tripartite" (there is only one such connection pattern). Let Z be the partition function of such configurations. Let a, b, c, d be boundary vertices that are pairwise connected in the connection pattern, and for any $I \subset \{a, b, c, d\}$ let Z_I be the partition function obtained by removing the boundary vertices in I. Then it has been proved by Jenne that:

$$ZZ_{abcd} = Z_{ab}Z_{cd} + Z_{ad}Z_{bc}.$$

This identity has interpretations in cluster algebras. Can one find a probabilistic application?

2.2 Scott Sheffield

Sheffield discussed two open problems.

- Are there some "nearly planar" maps for which dimer models can be analyzed (showing convergence of suitably defined height functions to the GFF, understanding double dimer paths, etc.)?
- Can one understand the different ways that weighting by a dimer partition function can change the law of the scaling limit of a random planar map model?

2.3 Richard Kenyon

For $n \geq 2$, let $\Delta_n = \{x_1, \dots, x_n \geq 0 \mid x_1 + \cdots + x_n = 1\}$. Then one has

$$\int_{\Delta_n} \frac{(x_1 \dots x_n)^{n-2}}{(x_1 \dots x_{n-1} + \cdots + x_2 \dots x_n)^n} \mathrm{dvol} = \frac{1}{(n-1)!}.$$

There is a convoluted way of proving this, related to counting of acyclic orientations of a line. Does one have a more straightforward way of computing this integral?

2.4 Nathanaël Berestycki

In several settings, it is known that the dimer model's height function converges to a GFF. For a discrete GFF h_N in some domain, it is known that the maximum is of order

$$c \log N + c' \log \log N + \xi$$

where ξ is some random variable. Can one get even the first term for the maximum of the dimer model's height function?

2.5 Nishant Chandgotia

Consider a model for which there exists two locally uniform ergodic Gibbs measures μ_1, μ_2, whose entropy satisfy $h(\mu_1) < h(\mu_2)$. Under some assumptions on the model, can one prove that there is another locally uniform ergodic Gibbs measure μ_3 such that $h(\mu_1) < h(\mu_3) < h(\mu_2)$? An interesting example is the case of proper 4-colorings of $\mathbb{Z}^3$.

3 Presentation Highlights

3.1 Leonid Petrov: From Yang-Baxter equation to Markov maps

We obtain a new relation between the distributions μ_t at different times $t \geq 0$ of the continuous-time TASEP (Totally Asymmetric Simple Exclusion Process) started from the step initial configuration. Namely, we present a continuous-time Markov process with local interactions and particle-dependent rates which maps the TASEP distributions μ_t backwards in time. Under the backwards process, particles jump to the left, and the dynamics can be viewed as a version of the discrete-space Hammersley process. Combined with the forward TASEP evolution, this leads to a stationary Markov dynamics preserving μ_t which in turn brings new identities for expectations with respect to μ_t. The construction of the backwards dynamics is based on Markov maps interchanging parameters of Schur processes, and is motivated by bijectivizations of the Yang-Baxter equation. We also present a number of corollaries, extensions, and open questions arising from our constructions.

3.2 Michael Aizenman: A Quantum Dimerization Phenomenon and the self dual F-K Random Cluster Models

Unlike classical antiferromagnets, quantum antiferromagnetic systems exhibit ground state frustration effects even in one dimension. A case in point is a quantum spin chain with the interaction between neighboring S-spins given by the projection on the two-spins singlet state. This 1D quantum system's ground state bears a close analogy to the self dual 2D Fortuin-Kasteleyn random cluster model, at $Q = (2S+1)^2$. The corresponding stochastic geometric representation has led to the dichotomy (Aiz-Nachtergale): for each S the ground state exhibits either (i) slow decay of spin-spin correlations (as in the Bethe solution of the Heisenberg $S = 1/2$ antiferromagnet) or (ii) dimerization, manifested in translation symmetry breaking. Drawing on the recent analysis of the phase transition of the FK models (by Duminil-Gagnebin-Harel-Manolescu-Tassion, and Ray-Spinka), we show that in the infinite volume limit for any $S > 1/2$ this $SU(2S+1)$ invariant quantum system has a pair of distinct ground states, each exhibiting spatial energy oscillations, and exponential decay of correlations.

(Joint work with H. Duminil-Copin and S. Warzel).

3.3 Vadim Gorin: Shift invariance for the six-vertex model and directed polymers

Based on joint work with Alexei Borodin and Michael Wheeler.

This work is about a simple-looking property of a variety of integrable probabilistic systems that includes stochastic vertex models, (1+1)d directed polymers in random media and last passage percolation with specific weights, as well as universal objects of the Kardar-Parisi-Zhang universality class – the KPZ equation and the Airy sheet. The property says that joint distributions of certain multi-dimensional observables in the system are unchanged under a shift of a subset of observation points.

It can be thought of as a far reaching generalization of the following known feature of the Brownian bridge: Fix $a < b$ and let $B(t)$, $0 \leq t \leq 1$, be a Brownian bridge such that $B(0) = a$ and $B(1) = b$. Let $\mathcal{L}_c$ denote the local time that $B(t)$ spends at level c. Then as long as $a \leq c \leq b$, the distribution of $\mathcal{L}_c$ does not depend on the choice of c.

While the above property of the invariance of Brownian local times under the shifts of c admits a bijective proof, we have not been able to find anything similar for the more complicated systems that we deal with. Instead, our proofs rely on much more advanced machinery of Yang-Baxter integrable vertex models.

Beyond intrinsic interest, the shift-invariance property yields explicit formulas for certain multi-dimensional distributions that were not accessible before. The basic idea is that shifts sometimes allow to reduce complicated configurations of observation points to simpler ones, for which exact expressions are already known.

3.4 Alessandro Giuliani: Universal height fluctuations and scaling relations in interacting dimer models

In this talk I will review the results on the universality of height fluctuations in interacting dimer models, obtained in collaboration with F. Toninelli and V. Mastropietro in a recent series of papers. The class of models of interest are close-packed dimers on the square lattice, in the presence of small but extensive perturbations that make them non-determinantal. Examples include the 6-vertex model close to the free-fermion point and the dimer model with plaquette interaction. By tuning the edge weights, one can impose a non-zero average tilt for the height function, so that the considered models are in general not symmetric under discrete rotations and reflections. It is well known that, in the determinantal case, height fluctuations in the massless (or 'liquid') phase scale to a Gaussian log-correlated field and their amplitude is a universal constant, independent of the tilt. Our main result is the following: when the perturbation strength is sufficiently small, log-correlations survive, with amplitude A that, generically, depends non-trivially and non-universally on the perturbation strength and on the tilt. Moreover, the amplitude A satisfies a universal scaling relation ('Haldane' or 'Kadanoff' relation), saying that it equals the anomalous exponent of the dimer-dimer correlation. The main steps and ideas of the proof are the following:

- The interacting partition function and generating function for dimer correlations is written as a Grassmann integral analogous to ϕ^4 theory in 2D; such integral is invariant under a local gauge transformation, related to the local conservation law of dimer number
- Such a Grassmann ϕ^4 theory can be studied by multiscale methods (fermionic Renormalization Group); the iteration is convergent provided that the sequence of relevant and marginal coupling constants stays bounded in the infrared; there are 4 such constants, and we have at disposal only 3 counterterms (fixing the slope of the height and the anisotropy between horizontal and vertical dimers) in order to adjust their initial data - In order to control the flow of the fourth running coupling constant (the effective quartic interaction) we compare it with the flow of an exactly solvable model for interacting fermions in $d = 1 + 1$, known as the Luttinger model, whose infrared behavior is the same as the one of the fermionic representation of interacting dimers
- The Luttinger model is solvable in a strong sense: exact formulas for correlations and critical exponents are available; the comparison between the fermionic representation of dimers and Luttinger in the infrared regime allows us to write the asymptotic behavior of dimer correlations in terms of the density-density and mass-mass correlations of the Luttinger model
- The comparison between the lattice Ward Identities for the dimer correlation functions (associated with the local gauge transformation mentioned above) and those for the Luttinger model allows us to relate the bare parameters of the Luttinger model with the dressed parameters of the dimer model; such a relation allows us to export some of the scaling relations known for the Luttinger model to the dimer context.

3.5 Amol Aggarwal: Universality for Lozenge Tiling Local Statistics

A salient feature of random tiling models is that the local densities of tiles can differ considerably in different regions of the domain, depending on the boundary data. Thus, a question of interest, originally mentioned by Kasteleyn in 1961 [4], is how the shape of the domain affects the local behavior of a random tiling. In this talk, we consider uniformly random lozenge tilings of essentially arbitrary domains. We outline a proof of the result from [1] that the local statistics of this model around any point in the liquid region of its limit shape are given by the infinite-volume, translation-invariant, extremal Gibbs measure of the appropriate slope. This was predicted by Cohn-Kenyon-Propp in 2001 [2].

Unlike many of the previous proofs applicable to special domains, our method does not make direct use of a Kasteleyn matrix. Instead, we proceed by locally comparing a uniformly random lozenge tiling of a given domain with an ensemble of Bernoulli random walks conditioned to never intersect. The benefit to the latter model is that its algebraic structure appears to be more amenable to asymptotic analysis than does the Kasteleyn matrix. In particular, under reasonably general initial data, its convergence of local statistics was recently analyzed by Gorin-Petrov [3]. We show that the tiling model and path model can be coupled locally around a vertex in the liquid region so that they coincide with high probability. The convergence of local statistics for the latter model then imply the same for the former.

Central to implementing this procedure is to establish a "local law" for the random tiling, which states that the associated height function is approximately linear on any mesoscopic scale. The proof of this local law proceeds through a multi-scale analysis, using an effective global law at each scale and deterministic estimates showing that the gradient of the global profile is approximately constant between scales.

[1] A. Aggarwal, Universality for Lozenge Tiling Local Statistics, preprint, `https://arxiv.org/abs/1907.09991`.

[2] H. Cohn, R. Kenyon, and J. Propp, A Variational Principle for Domino Tilings, J. Amer. Math. Soc. 14, 297346, 2001.

[3] V. Gorin and L. Petrov, Universality of Local Statistics for Noncolliding Random Walks, To appear in Ann. Prob., preprint, `https://arxiv.org/abs/1608.03243`.

[4] P. W. Kasteleyn, The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice, Physica 27, 12091225, 1961.

3.6 Richard Kenyon: Darboux integrability and gradient models

This is joint work with Istvan Prause and is based on previous work with Jan de Gier and Sam Watson.

The 5-vertex model is a special case of the 6-vertex model and a generalization of the lozenge dimer model. We show how to compute the surface tension and limit shapes in the 5-vertex model via an explicit computation using the Bethe Ansatz technique.

One feature of the five vertex model is that the Euler-Lagrange equation for the surface tension minimizers has an explicit solution in terms of arbitrary analytic function inputs. This is known as "Darboux integrability".

We showed that in fact any gradient model (variational problem for a function in two variables whose surface tension only depends on the gradient) is Darboux integrable by the same technique, on condition that the determinant of the Hessian of sigma is the fourth power of a harmonic function (harmonic in the underlying conformal coordinate).

This allowed us to extend our results on the 5-vertex model to a staggered weight 5-vertex model in which the weights are of isoradial type.

3.7 Marcin Lis: Spins, percolation and height functions

To highlight certain similarities in combinatorial representations of several well known two-dimensional models of statistical mechanics, we introduce and study a new family of models which specializes to these cases after a proper tuning of the parameters. To be precise, our model consists of two independent standard Potts models, with possibly different numbers of spins and different coupling constants (the four parameters of the model), defined jointly on a graph embedded in a surface and its dual graph, and conditioned on the event that the primal and dual interfaces between spins of different value do not intersect. We also introduce naturally related height function and bond percolation models, and we discuss their basic properties and mutual relationship. As special cases we recover the standard Potts and random cluster model, the 6-vertex model and loop $O(n)$ model, the random current, double random current and XOR-Ising model.

3.8 Alexander Glazman: Six-vertex and Ashkin-Teller models: order/disorder phase transition

Ashkin-Teller model is a classical four-component spin model introduced in 1943. It can be viewed as a pair of Ising models τ and τ with parameter J that are coupled by assigning parameter U for the interaction of the products $\tau\tau$ at every two neighbouring vertices. On the self-dual curve $\sinh 2J = e^{-2U}$, the Ashkin-Teller model can be coupled with the six-vertex model with parameters $a = b = 1$, $c = \coth 2J$ and is conjectured to be conformally invariant. The latter model has a height-function representation. We show that the height at a given face diverges logarithmically in the size of the domain when $c = 2$ and remains uniformly bounded when $c > 2$. In the latter case we obtain a complete description of translation-invariant

Gibbs states and deduce that the Ashkin-Teller model on the self-dual line exhibits the following symmetry-breaking whenever $J < U$: correlations of spins τ and τ decay exponentially fast, while the product $\tau\tau$ is ferromagnetically ordered. The proof uses the Baxter-Kelland-Wu coupling between the six-vertex and the random-cluster models, as well as the recent results establishing the order of the phase transition in the latter model. However, in the talk, we will focus mostly on other parts of the proof:

- description of the height-function Gibbs states via height-function mappings and T-circuits,
- coupling between the Ashkin-Teller and the six-vertex models via an FK-Ising-type representation of these two models.

(this is joint work with Ron Peled)

3.9 Yinon Spinka: Discontinuity of phase transition of the planar random cluster model for q larger than 4: a short proof

The random-cluster model is by now a well-known dependent percolation model, which is also closely related to the Potts model. We consider here the random-cluster model on the square lattice $\mathbb{Z}^2$. An important quantity of interest in this model is the probability $\theta^{\mathrm{f/w}}(p,q)$ that the origin belongs to an infinite cluster under the free/wired random-cluster measure. For any $q \geq 1$, the model undergoes a phase transition at $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ in the sense that $\theta^{\mathrm{f}}(p,q) = \theta^{\mathrm{w}}(p,q) = 0$ for all $p < p_c$ and $\theta^{\mathrm{f}}(p,q) = \theta^{\mathrm{w}}(p,q)$ for all $p > p_c$. The behavior at the critical parameter p_c is of particular interest. For example, the two functions $\theta^{\mathrm{f/w}}(p,q)$ are always continuous at every $p \neq p_c$, whereas they are continuous at $p = p_c$ precisely when $\theta^{\mathrm{w}}(p_c,q) = 0$; in this case, the phase transition is said to be continuous.

Baxter [1] conjectured that the phase transition is continuous for $1 \leq q \leq 4$ and discontinuous for $q > 4$. This was recently verified in a combination of two beautiful papers: the regime $1 \leq q \leq 4$ was proved by Duminil-Copin, Sidoravicius and Tassion [3] and the regime $q > 4$ by Duminil-Copin, Gagnebin, Harel, Manolescu and Tassion [2]. We reprove the latter discontinuity of phase transition when $q > 4$ via a short probabilistic proof.

The original proof of discontinuity given in [2] relies on an analysis of the so-called Bethe ansatz aimed at computing the eigenvalues of a transfer matrix for the six-vertex model. Such an approach has the advantage of yielding a precise expression for the exponential rate of decay of the probability (under the free random-cluster measure) that the origin is connected to a far away point.

Our proof, while also exploiting the connection with the six-vertex model, does not rely on the Bethe ansatz. Instead, our proof is based on softer arguments (in particular, it does not require a computation of the correlation length) and only uses very basic properties of the random-cluster model (for example, we do not even need the Russo–Seymour–Welsh machinery developed recently in [3]).

Joint work with Gourab Ray.

[1] Rodney James Baxter, *Solvable eight-vertex model on an arbitrary planar lattice*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **289** (1978), no. 1359, 315346.

[2] Hugo Duminil-Copin, Maxime Gagnebin, Matan Harel, Ioan Manolescu, and Vincent Tassion, *Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$*, arXiv preprint arXiv:1611.09877 (2016).

[3] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion, *Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$*, Communications in Mathematical Physics **349** (2017), no. 1, 47107.

3.10 Zhongyang Li: Limit shape and height fluctuations of perfect matchings on square-hexagon lattices

We study asymptotics of perfect matchings on a large class of graphs called the contracting square-hexagon lattice,which is constructed row by row from either a row of a square grid or a row of a hexagonal lattice. We assign the graph periodic edge weights with period $1 * n$, and consider the probability measure of perfect matchings in which the probability of each configuration is proportional to the product of edge weights. We show that the partition function of perfect matchings on such a graph can be computed explicitly by a Schur function depending on the edge weights. By analyzing the asymptotics of the Schur function, we then prove the Law of Large Numbers (limit shape) and the Central Limit Theorem (convergence to the Gaussian free field) for the corresponding height functions. We also show that the distribution of certain type of dimers near the turning corner is the same as the eigenvalues of Gaussian Unitary Ensemble,and explicitly study the curve separating the liquid region and the frozen region for certain boundary conditions.

3.11 Sanjay Ramassamy: Dimers and circle patterns

In joint work with R. Kenyon, W. Y. Lam and M. Russkikh, we establish a correspondence between dimer models on planar bipartite graphs and centers of circle patterns. Circle patterns are a class of objects coming from discrete differential geometry, where they serve as discrete conformal maps.

Let G be a planar bipartite graph which is realized as a circle pattern, meaning that every vertex is mapped to a point in the plane in such a way that every face admits a circumcircle. Assume additionally that the realization of the dual graph of G (induced by mapping every dual vertex to the center of the circle corresponding to that face) is an embedding. Then one can assign a complex number to every edge of the bipartite graph, this complex number being the vector connecting the two centers lying on each side of that edge, such that the black endpoint of the edge lies to the left of the vector. This assignment of complex numbers to edges satisfies the Kasteleyn condition, a condition required on the entries of the Kasteleyn matrix used to compute the partition function and the correlations of the dimer model on G. Such a construction generalizes the isoradial case (all the circles have the same radius) proposed by Kenyon in 2002.

Conversely, let G be a planar bipartite graph equipped with edge weights satisfying the Kasteleyn condition. We ask whether there exist gauge equivalent edge weights (meaning weights inducing the same Boltzmann probability measure on dimer coverings of G) coming from an embedding as circle centers. Such a choice of edge weights is called a Coulomb gauge, because of the fact that it has zero divergence (the sum of the edge weights around any given vertex of G vanishes). We provide a positive answer to that question in the cases of finite planar graphs with outer face of degree four and of infinite planar graphs which are periodic in two directions. Given a choice of boundary conditions, Boltzmann measures for the dimer model on an unweighted planar bipartite graphs with outer face of degree 4 are in one-to-two correspondence with Coulomb gauges. For an infinite periodic graph, Coulomb gauges are in one-to-one correspondence with liquid ergodic Gibbs measures.

One application of this correspondence is in the field of geometry and dynamics. This correspondence identifies a local move on circle patterns called the Miquel move to a local move on dimers models called the urban renewal. This enables us to identify Miquel dynamics, a discrete-time dynamical system on the space of square-grid circle patterns, to the Goncharov-Kenyon dynamics on dimer models, from which the integrability of Miquel dynamics follows.

One can also hope to apply this correspondence to statistical mechanics, namely to the study of scaling limits for the dimer model. It is expected that circle center embeddings provide the right geometric setting to study general dimer models. One indication of this comes from the fact that they generalize the Tutte embedding adapted to the study of spanning trees and Chelkak's s-embeddings adapted to the study of the Ising model. Another indication comes from a general framework proposed by Chelkak-Laslier-Russkikh for obtaining the fluctuations as a Gaussian free field from the embedding as circle centers, as was explained in the later talks by Laslier and Chelkak. To illustrate this general framework, in a ongoing joint project with D. Chelkak, we are attempting to rederive via this method the conformal structure used to the define the Gaussian free field in the case of the Aztec diamond, where exact computations can be made.

3.12 Benoit Laslier and Dmitry Chelkak: Perfect t-embeddings of bipartite planar graphs and the convergence to the GFF - I & II

We discuss a concept of 'perfect t-embeddings, or 'p-embeddings', of weighted bipartite planar graphs. (T-embeddings also appeared under the name Coulomb gauges in a recent work of Kenyon, Lam, Ramassamy and Russkikh.) We believe that these p-embeddings always exist and that they are good candidates to recover the complex structure of big bipartite planar graphs carrying a dimer model. To support this idea, we first develop a relevant theory of discrete holomorphic functions on t-embeddings; this theory unifies Kenyon's holomorphic functions on T-graphs and s-holomorphic functions coming from the Ising model.

Further, given a sequence of (abstract) planar graphs G_n and their p-embeddings T_n onto the unit disc D, assume that (i) the faces of T_n satisfy certain technical assumptions in the bulk of D; (ii) the size of the associated origami maps O_n tends to zero as n grows (again, on each compact subset of D). We prove that (i)+(ii) imply the convergence of the fluctuations of the dimer height functions on G_n (provided that these graphs are embedded by T_n), to the GFF on the unit disc D equipped with the standard complex structure. Though this is not fully clear at the moment, we conjecture that the origami maps O_n are always small in absence of frozen regions and gaseous bubbles, so our theorem can be eventually applied to all such cases. Moreover, the same techniques are applicable in the situation when the limit of the origami maps arising from a sequence of p-embeddings is a Lorenz-minimal surface, in this situation one eventually obtains the GFF in the conformal parametrization of this surface.

In a related joint work with Sanjay Ramassamy we argue that such a Lorenz-minimal surface indeed arises in the case of classical Aztec diamonds; a general conjecture is that this should 'always' be the case due to a link between p-embeddings and a representation of the dimer model in the Plücker quadric. Time permitting, we also indicate how the theory of t-holomorphic functions specifies to the Ising case and discuss related results on conformal invariance of the Ising model as well as a more general perspective.

3.13 Nathanael Berestycki: The dimer model on Riemann surfaces

Temperley's bijection is a powerful tool for the study of the dimer model in the simply connected setting. This bijection relates the behaviour of a pair of dual uniform spanning trees to the dimer model on the graph obtained by superposing the planar and dual graph together with medial vertices: more specifically the height differences in the dimer model on this graph is equal to the winding of branches in either dual or primal spanning trees. Using this connection and its powerful generalisation by Kenyon and Sheffield (to the setting of T-graphs), we were able in a previous work to derive robust proofs of convergence of the height function in the simply connected setting to the Gaussian free field (GFF): in the scaling limit, the branches of the spanning tree become the flow line of the GFF, in other words the GFF and continuum spanning trees are related through imaginary geometry.

In this work we consider the case of Riemann surfaces, which is much harder to study via determinantal methods than it is in the simply connected case. Our goal is to prove the existence, universality and conformal invariance of the scaling limit of the height one-form in the dimer model. Consider the superposition of a graph and its dual (as well as medial vertices) embedded on a given Riemann surface M with finitely many holes and handles. Removing the correct number of medial vertices (which can be computed via Euler's formula), we show that the resulting graph is dimerable. These removed medial vertices can be thought of as punctures in the surface. We can apply Temperley's bijection since it is locally defined, but the resulting object will not be a pair of dual spanning trees. Instead, there are global topological constraints and we call the resulting (locally tree-like) structure Temperleyan forests (and their dual). A Temperleyan forest may contain cycles but only nontrivial ones, so that they are reminiscent of a simpler object: the Cycle-Rooted Spanning Forest (CRSF). In fact, we show that a CRSF is Temperleyan if and only if the branches emanating from every puncture decompose the surface into disjoint annuli; on a torus or an annulus every CRSF is Temperleyan (up to a choice of orientation of each cycle).

Furthermore, the height-form of the dimer model is, as in the simply connected case, closely related to the behaviour of the Temperleyan forest. Indeed, adopting the Fuchsian point of view on Riemann surfaces (in which the surface is represented as a polygon with certain periodic boundary conditions), we show that height differences can be identified with the winding of branches of the Temperleyan forest (as in the simply connected setting) plus some global topological corrections, which correspond to the fact that two points may not belong to the same component of the forest (so that one needs to "jump" over components). Given this result, we can apply ideas from our earlier work to show that if the Temperleyan forest admits a scaling limit, then so does the height-form: this is because winding is in some sense well behaved, whereas the global topological correction terms are easily handled.

Finally, on a torus or an annulus, a Temperleyan forest is equivalent to a CRSF (up to a Radon-Nikodym factor proportional to $2^{\#\text{cycles}}$). Since a CRSF can be sampled through a version of Wilson's algorithm, we show that (on any surface) CRSFs have a universal and conformally invariant scaling limit. Consequently the same holds for Temperleyan forests. Putting these results together, we obtain the scaling limit of the dimer height-form on the torus and the annulus. (A generalisation of this result to arbitrary surfaces is ongoing work.) On the torus, Dubedat showed that in the double isoradial case the limit was a compactified GFF, so our universality result implies this is always the case (solving a conjecture of Dubedat-Gheissari).

Joint work with Benoit Laslier (Paris) and Gourab Ray (Victoria).

3.14 Kurt Johansson: On the rough-smooth interface in the two-periodic Aztec diamond

The two-periodic Aztec diamond is a certain random tiling model. It can also be thought of as a dimer model or perfect matching on a certain bipartite graph. In models of this type it is possible to have three types of local dimer patterns, or limiting Gibbs measures, called frozen (solid), rough (liquid) ot smooth (gas). The double Aztec diamond is probably the simplest model where we can get a coexistence of all three phases so that we have, asymptotically, interfaces between the frozen and rough phases, as well as between the rough and smooth phases. The purpose of the work I report on is to understand the (local) geometry of the rough-smooth interface.

At the frozen-rough interface we have a well-defined boundary path at the discrete the level which converges to the Airy process after appropriate rescaling. The Airy process is a universal stochastic process that appears in many contexts in particular in random matrix theory and in random growth models. It is conjectured that at a typically at the interface between a rough and a frozen phase in a large class of random tiling models, the boundary process converges to the Airy process.

At the rough-smooth smooth boundary the situation is more complicated although here also we expect to have an Airy process. It is not obvious which geometric structure actually converges to the Airy process. The frozen boundary is clear, it is the place where we first see a change from the regular pattern. It is less clear exactly how one should define the boundary at the discrete level at the rough-smooth interface. At this boundary we see both local and long range structures and looking just locally, we can not tell if we are encountering a long range path or just a local structure (a loop). In joint work with Sunil Chhita and Vincent Beffara, we define this boundary precisely. We are not able to show that there is actually a last path that converges to the Airy process, but we are able to define a certain (signed) counting measure which in a sense counts the number of paths between in prescribed intervals, and show that this random measure converges to the Airy kernel point process.

3.15 Patrik Ferrari: Time-time correlation for the North polar region of the Aztec diamond

This talk is based on the papers [1] with H. Spohn and [2] with A. Occelli. We consider the boundary of the North polar region of the Aztec diamond. The interface between the random region and the frozen one is what we call the height function. Due to the shuffling algorithm, our model is a discrete time Markov chain, where time equals the size of the Aztec diamond.

We are interested in the time correlations of the height function. In particular we determine the limiting behavior of the covariance of the height function. When the two times are macroscopically close, that is, we consider time τN and time N with $1-\tau$ small, the first order correction of the time-time covariance is universal and given in term of the variance of the Baik-Rains distribution function (which is the stationary limiting distribution in KPZ models).

The result is proven in [2] in the language of last passage percolation with exponential random variables. However, due to the well-known link between Aztec diamond and discrete time TASEP with parallel update, which in turns is equivalent to a last passage percolation with geometric random variables, the same result will holds also for the Aztec diamond case.

[1] P.L. Ferrari and H. Spohn, On time correlations for KPZ growth in one dimension, SIGMA 12 (2016), 074.

[2] P.L. Ferrari and A. Occelli, Time-time covariance for last passage percolation with generic initial profile, Math. Phys. Anal. Geom. 22 (2019), 1.

3.16 Beatrice De Tiliere: Elliptic dimers and genus 1 Harnack curves

We consider the dimer model on a bipartite periodic graph with elliptic weights introduced by Fock. The spectral curves of such models are in bijection with the set of all genus 1 Harnack curves. We prove an explicit and local expression for the two-parameter family of ergodic Gibbs measures and for the slope of the measures.

This is work in progress with Cédric Boutillier and David Cimasoni.

3.17 Martin Tassy: Uniqueness of the limiting profile for monotonic Lipschitz random surfaces

For dimers and other models of random surfaces, limit shapes appear when boundary conditions force a certain response of the system. The main mathematical tool to study these responses is a variational principle which states that the limiting profile of the system must maximizes the integral of an entropy function often named surface tension. As a consequence, the strict convexity of the surface tension plays a crucial role as it forces the asymptotic profile which maximizes this integral to be unique. In this talk we will show that all models of Lipschitz random surfaces which are stochastically monotonic must have a strictly convex surface tension (joint with Piet Lammers).

3.18 Scott Sheffield: Laplacian determinants and random surfaces

My talk explored the ways that dimer models and other statistical physics models are related to Laplacian determinants, both on the discrete level and on the continuum level.

In particular, I recalled the geometric meaning of the so-called zeta-regularized determinant of the Laplacian, as it is defined on a compact surface, with or without boundary. Using an appropriate regularization, we found that a Brownian loop soup of intensity c has a partition function described by the $(-c/2)^{\text{th}}$ power of the determinant of the Laplacian. In a certain sense, this means that decorating a random surface by a Brownian loop soup of intensity c corresponds to weighting the law of the surface by the $(-c/2)^{\text{th}}$ power of the determinant of the Laplacian.

I then introduced a method of regularizing a unit area LQG sphere, and showing that weighting the law of this random surface by the $(-c/2)^{\text{th}}$ power of the Laplacian determinant has precisely the effect of changing the matter central charge from c to c'. Taken together with the earlier results, this provided a way of interpreting an LQG surface of matter central charge c as a pure LQG surface decorated by a Brownian loop soup of intensity c.

This talk was based on joint work with Morris Ang, Minjae Park, and Joshua Pfeffer.

3.19 Paul Melotti: The eight-vertex model via dimers

The eight-vertex model is an ubiquitous description that generalizes several spin systems, "ice-type" six-vertex models, and is related to Ashkin-Teller model, XYZ spin chains, and others. In a special "free-fermion" regime, it is known since the work of Fan, Lin, Wu in the late 60s that the model can be mapped to non-bipartite dimers, while in the six-vertex case the graph becomes bipartite. In this talk I show a relation between these non-bipartite dimers and a couple of bipartite dimers, that correspond to two different free-fermion 6V-models. More precisely, for any free-fermion 8V-model on a quadrangulation whose faces are colored black and white, there exists two free-fermion 6V-models that are in relation with the original 8V-model. The relation takes several forms:

- at the level of characteristic polynomials, that of the 8V-model is the product of those of the 6V-ones. Hence the former is the union of two Harnack curves;
- at the level of partition functions, the square of the 8V-one is the product of the 6V-ones;
- at the probabilistic level, the symmetric difference of two independent (identically distributed) 8V-configuration has the same distribution as the symmetric difference of two independent (differently distributed) 6V-configurations;

- at the level of Kasteleyn matrices, there is a simple relation between the inverse operators of all these models.

These can be proved using the formalism of order-disorder variables in a framework close to that of Dubédat.

This construction can be applied in an isoradial, Z-invariant setting, *i.e.* a regime of the model on the dual of a lozenge graph such that the star-triangle transformation is automatically satisfied. The relation, in combination with works of Boutillier, de Tilière and Raschel, yields the existence of an ergodic Gibbs measure, and exact, local formulas for correlations.

4 Scientific Progress Made

This program and hospitality of BIRS made it possible for several fruitful discussions and collaborations. We highlight a few here.

One point of controversy arose from a question of Sheffield (Section 2.2). It is common knowledge that every statistical physics model has an inherent central charge associated with them which governs the universality class of the model. The question was: is there any natural such class for the dimer model? For example, it was known since the time of Fisher that the Ising model can be mapped to a dimer model. Later, Dubedat showed that two Ising models can be mapped to a dimer model, a phenomenon known as *Bosonization*. On the other hand, Temperley's bijection maps Uniform spanning trees to dimer model. All these models are supposed to have different central charges. This sparked some informal debate on what is a 'natural' central charge that can be associated to the dimer model, and the general consensus was that dimer model has certain non-universal nature which impedes the assignment of such a universality class to the model. Various aspects of this question remain a matter of speculation and debate.

Sheffield asked another question regarding the analysis of dimer models on simplest non-planar domains, for example, one can add some 'pipes' in $\mathbb{Z}^2$. This sparked some interest and several possible suggestions were made. In the end it was unclear if a non-trivial limit can be achieved which is away from the gaseous phase of the model.

Another highlight was regarding the question of Berestycki concerning the maxima of the height function of the dimer model. After the proposal of the question, several suggestions were made by experts on how to progress with this question. In some cases, dimer height differences have the same law as a sum of independent Bernoulli variables; Dubedat suggests this might be a good starting point to study dimer height maxima.

5 Outcome of the Meeting

It was a general feeling during the meet that all the talks were of *very* high standard and engaging. Some of the comments regarding the meet by the participants is that it is quite rare for so many experts in the field to congregate for a specialized workshop like the one made possible here. Overall, this opened the doorway for some fruitful collaborations, raised some pertinent questions, and opened up the possibilities for future research directions for many of the participants.

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