Embezzlement of Entanglement

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They also gave some estimates on the dimensions of $\mathcal{R}_A$ and $\mathcal{R}_B$ needed to carry out this process as a function of ϵ.

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Note that

$$(U_A \otimes I_{\mathcal{R}_B} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes I_{\mathcal{R}_A} \otimes U_B) \\ = U_A \otimes U_B = \\ (I_{\mathcal{H}_A} \otimes I_{\mathcal{R}_A} \otimes U_B)(U_A \otimes I_{\mathcal{R}_B} \otimes I_{\mathcal{H}_B}).$$

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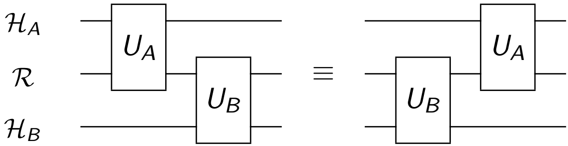

Theorem (Cleve-Liu-P, Harris-P)

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional. Given any unit vector $\phi = \sum_{i,j} \alpha_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exists a Hilbert space $\mathcal{R}$, a unit vector $\psi \in \mathcal{R}$, unitaries

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Briefly, catalytic production of entanglement is possible in the commuting operator model.

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Lemma

$(U_A \otimes id_B)$ commutes with $(id_A \otimes U_B)$ if and only if $U_{i,j} V_{k,l} = V_{k,l} U_{i,j}$ and $U_{i,j}^* V_{k,l} = V_{k,l} U_{i,j}^*$ for all i, j, k, l.

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Thus, we see that having commuting operator frameworks as above is exactly the same as having operator matrices $U_A = (U_{i,j})$ and $U_B = (V_{k,l})$ that yield unitaries and whose entries pairwise *-commute.

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Thus, a representation of $U_{nc}(n) \otimes_{max} U_{nc}(m)$ corresponds to operators $U_{i,j}, V_{k,l}$ where the $U_{i,j}$'s *-commute with the $V_{k,l}$'s such that $(U_{i,j})$ and $(V_{k,l})$ are unitary operator matrices.

Theorem (Cleve-Liu-P, Harris-P)

Perfect embezzlement of a state $\phi = \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_{i,k} |i\rangle \otimes |k\rangle$ *is possible in a commuting operator framework if and only if there is a state* s *on* $U_{nc}(n) \otimes_{max} U_{nc}(m)$ *satisfying* $s(u_{i,1} \otimes v_{k,1}) = \alpha_{i,k}$.

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The approximate embezzlement results yield states on $U_{nc}(n) \otimes_{min} U_{nc}(m)$ that converge to a state on $U_{nc}(n) \otimes_{min} U_{nc}(m)$ satisfying the above equations, and hence the desired state on $U_{nc}(n) \otimes_{max} U_{nc}(m)$.

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The occurrence of min and max tensors in different places lead me to wonder what is their relationship? Maybe they are the same?

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Theorem (Harris)

The following are equivalent.

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The equivalence of the first three, is the analogue of Kirchberg's theorem relating Connes to tensor products of free group C*-algebras. The equivalence of the first and last is the analogue of the results of Junge, Navascues, Palazuelas, Perez-Garcia, Scholz, Werner and separately, Ozawa, relating CEP to Tsirelson's problems.

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The set $B_{qc}(n,m)$ is defined similarly except we replace the tensor product of two spaces by a single space and instead demand that the $U_{i,j}$'s *-commute with the $V_{k,l}$'s.
Here are some of the things that we know/don't know about these sets.

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Next we give an operational meaning to these sets.

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Each protocol yields a matrix $X \in B_q(n, m)$(resp. $B_{qc}(n, m)$) such that the bias of that protocol is $Re(Tr(HX))$.

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- *CEP is true iff* $bias_q(G) = bias_{qc}(G), \forall G$.

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Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional. If $\psi = \sum_{i,j} \beta_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and its highest Schmidt coefficient satisfies $\lambda_1 \leq \sqrt{\frac{2}{3}}$,

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$$\|U_A \otimes U_B(|0\rangle \otimes \psi \otimes |0\rangle) - \sum_{i,j} \beta_{i,j}|i\rangle \otimes \psi \otimes |j\rangle\| \geq \frac{2}{3}(3 - 2\sqrt{2})$$

and this bound is independent of the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$.

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$$(U_A \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes U_B)(|0\rangle \otimes \psi \otimes |0\rangle) = \sum_{i,j} \beta_{i,j} |i\rangle \otimes \psi \otimes |j\rangle \simeq \psi \otimes \psi.$$

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Thanks!