Asymptotic performance of port-based teleportation

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Joint work with

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Resource conversion in teleportation

- **Standard teleportation:** [Bennett et al. 1993]

$$[qq] + 2[c \to c] \geq [q \to q]$$

- **Port-based teleportation:** [Ishizaka and Hiroshima 2008]

$$N[qq] + (\log N)[c \to c] \geq [q \to q]$$

This is worse — why care?

Because port-based teleportation has unitary covariance!

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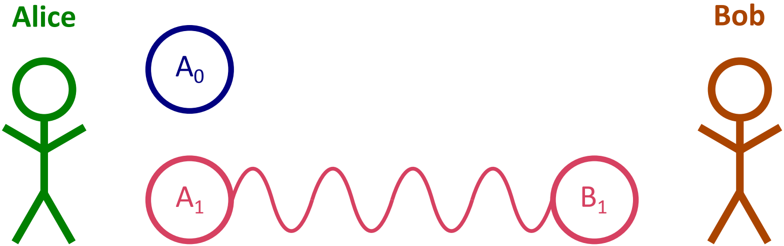
Standard teleportation protocol



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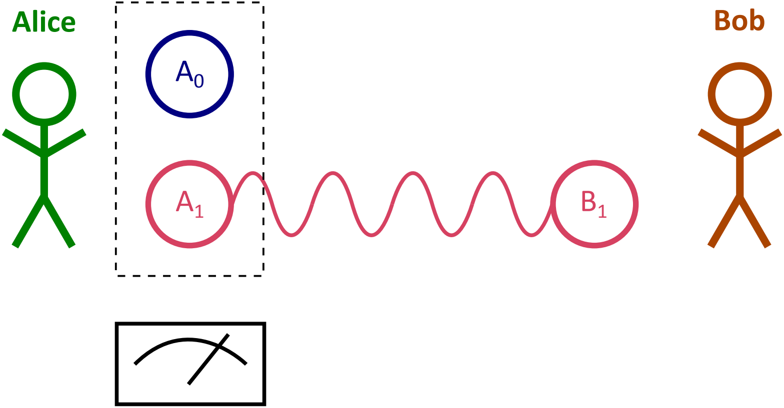

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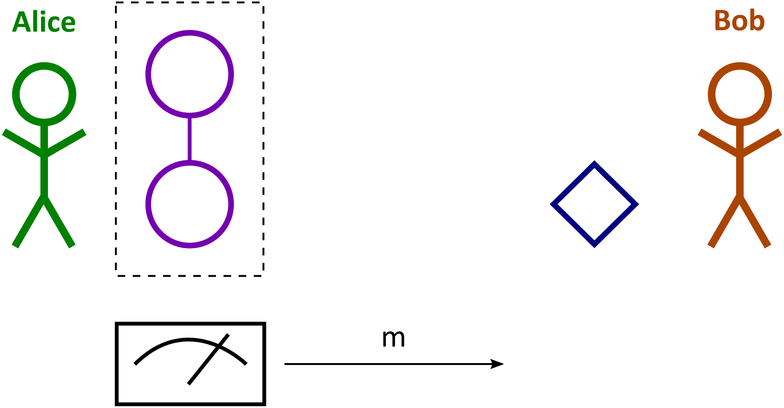

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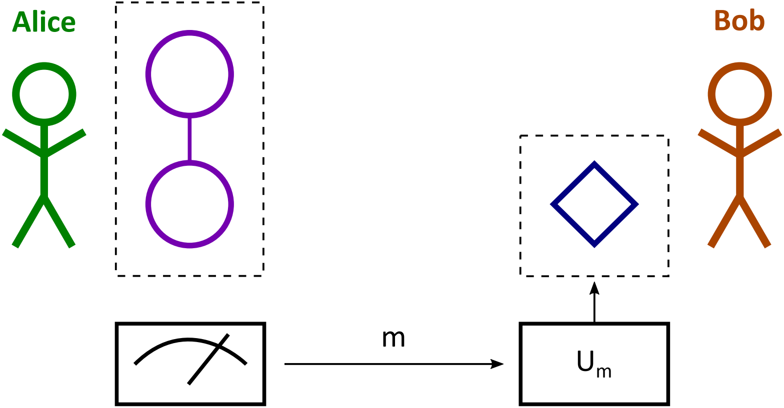

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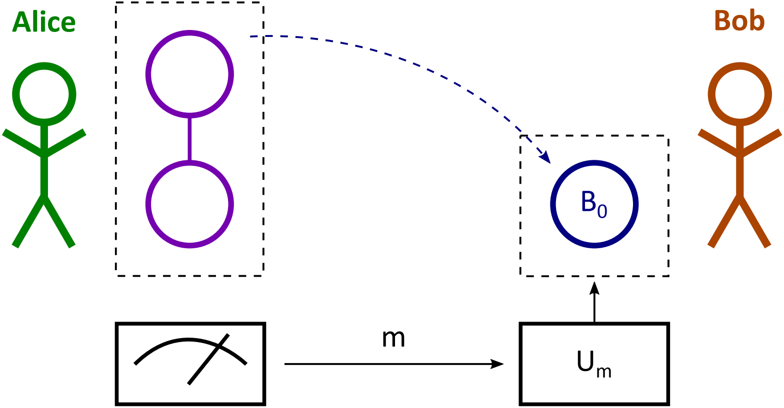

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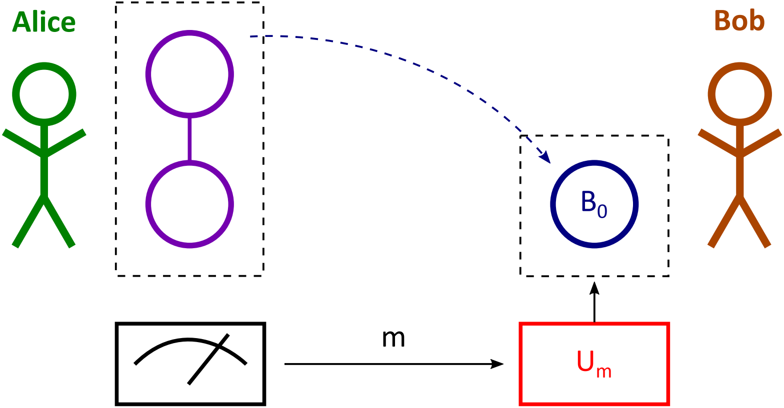
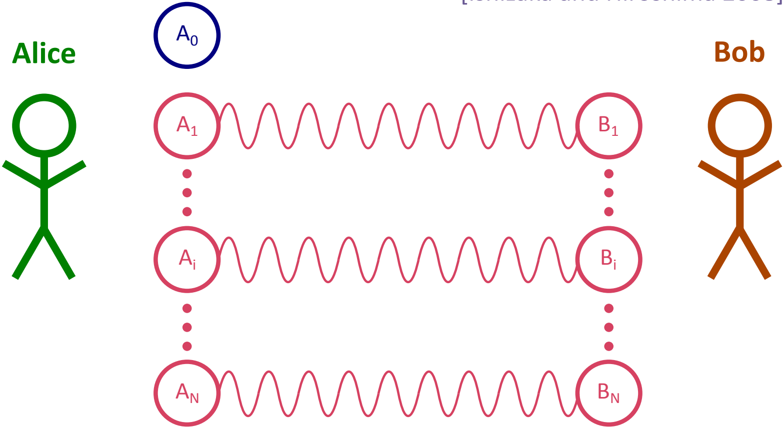


Port-based teleportation

[Ishizaka and Hiroshima 2008]



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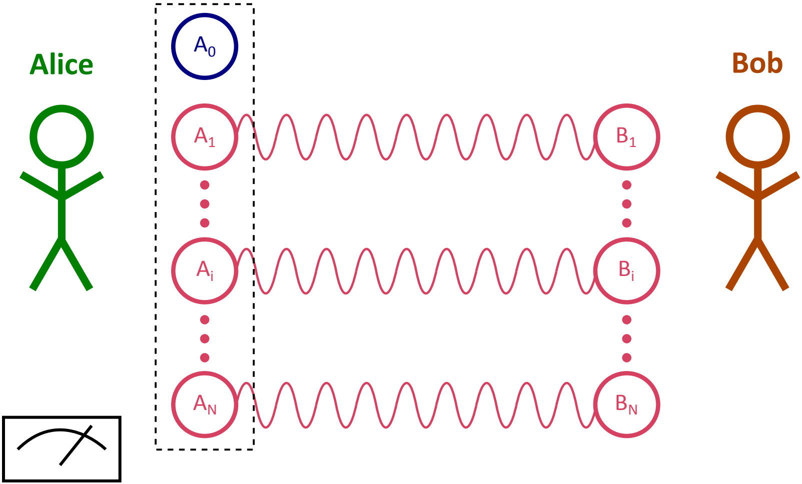

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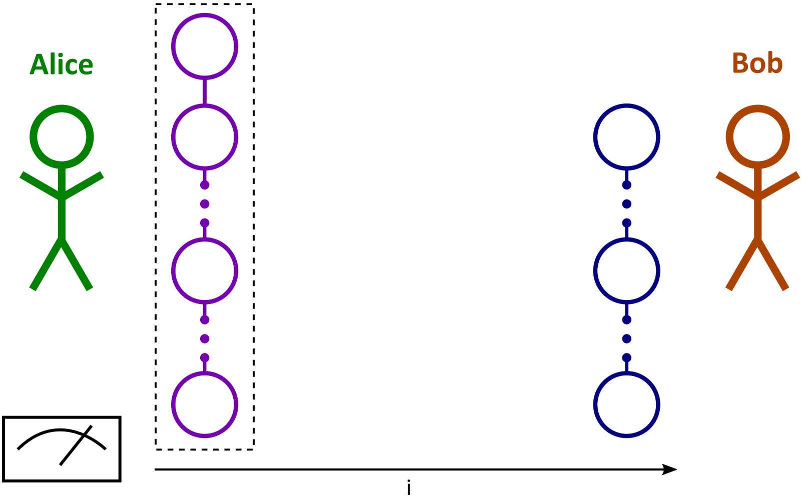

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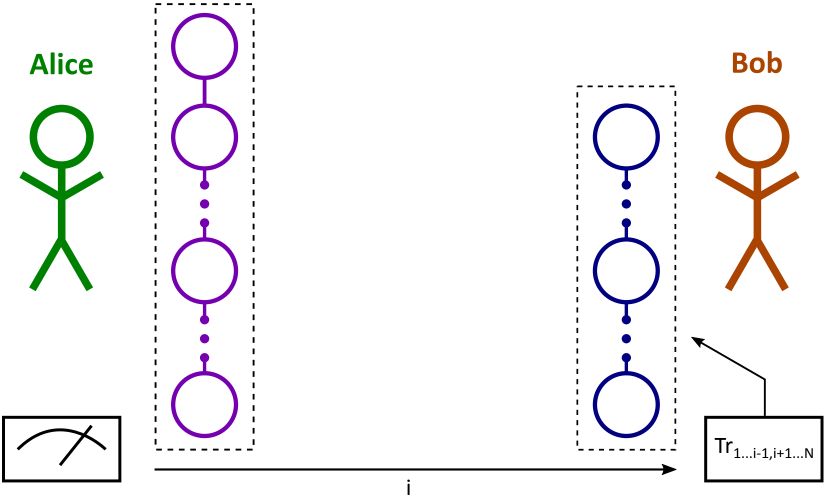

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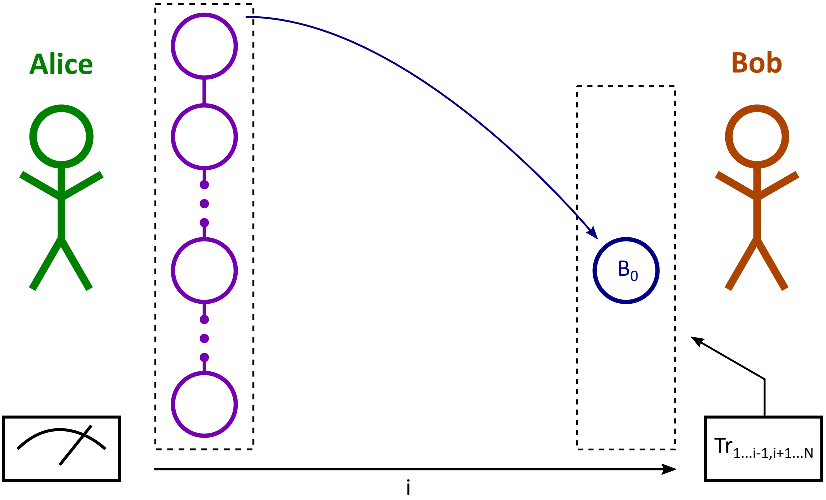

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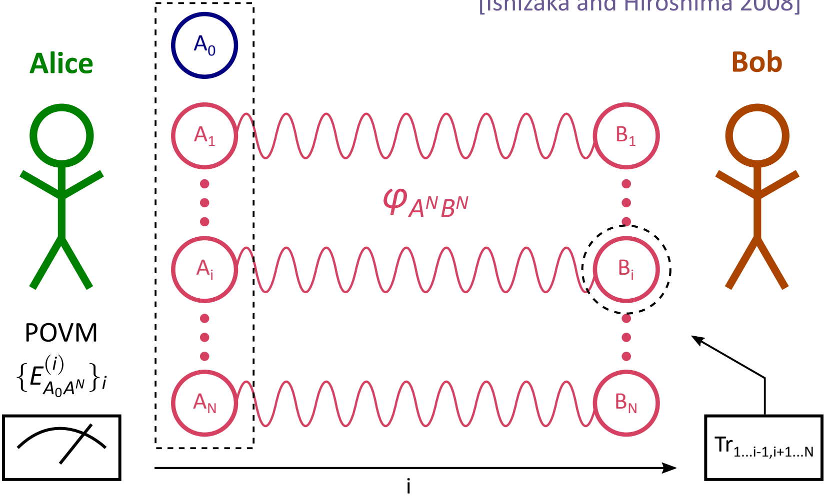

Port-based teleportation

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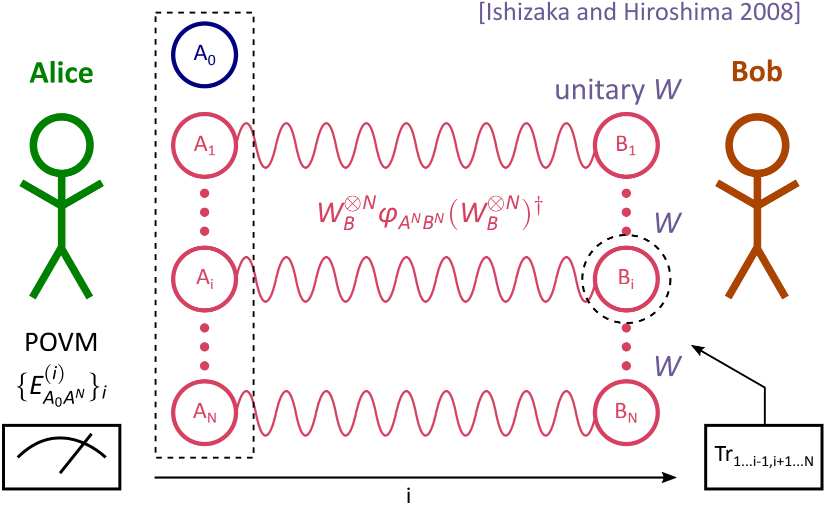

Why is PBT interesting?

- Partial trace commutes with $W^{\otimes N}$: PBT is **unitarily covariant**.
- PBT enables instantaneous non-local quantum computation (INQC). [Beigi and König 2011]
- INQC can be used to break position-based cryptography. [Buhrman et al. 2014]

Caveat

Unitary covariance leads to the fact that

perfect PBT is impossible with finite resources.

[Nielsen and Chuang 1997; Ishizaka and Hiroshima 2008]

Variants of PBT

Deterministic PBT

Protocol always yields final state that **approximates** target state.

Probabilistic PBT

Protocol yields **exact** target state with certain probability.

- **Unitary covariance:** Perfect PBT impossible with finite resources.
- **Goal of this talk:** Understand symmetries of PBT and determine asymptotic performance of PBT protocols.

Outline

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Deterministic PBT

- In deterministic PBT the protocol always yields a final state as an approximation to the target state.
- Hence, PBT protocol implements qudit channel Λ that **simulates ideal channel**.
- **Figure of merit:** entanglement fidelity

$$F_d = F(\Lambda, \mathrm{id}) = \langle \Phi^+_{A'A} | (\mathrm{id} \otimes \Lambda)(\Phi^+_{A'A}) | \Phi^+_{A'A} \rangle$$

- For PBT the diamond norm distance is exactly equivalent to F_d:

$$\| \mathrm{id} - \Lambda \|_\diamond = 2(1 - F_d).$$

[Pirandola et al. 2018]

Deterministic PBT and state discrimination

- Deterministic PBT is equivalent to **state discrimination** of the uniformly drawn states [Ishizaka and Hiroshima 2009]

$$\omega^{(i)}_{A^N B} = \mathrm{Tr}_{B_i^c}\, \varphi_{A^N B^N}.$$

- Success probability q of discriminating between $\omega^{(i)}$:

$$q = \frac{d^2}{N} F_d.$$

- Suggests **pretty good measurement** (PGM) as POVM.
- Further protocol simplification: $|\varphi\rangle = \mathrm{EPR}^{\otimes N}$
- We call $(\mathrm{EPR}^{\otimes N}, \mathrm{PGM})$ the **standard protocol**.

Probabilistic PBT

- Probabilistic PBT yields the **exact target state** with **success probability** p_d and aborts otherwise.
- Extended POVM $E_{\text{prob}} = \{E^{(i)}\}_{i=0}^{N}$, where $E^{(0)}$ corresponds to abortion of the protocol.
- Probabilistic PBT is a **special case of deterministic PBT**.
  (Send random port when getting outcome "0".)
- Again: consider special case where $|\varphi\rangle = \text{EPR}^{\otimes N}$.
- We call $(\text{EPR}^{\otimes N}, E_{\text{prob}})$ the **EPR protocol**.
  (POVM E_{prob} is now optimized over.)

Existing results: optimal performance of PBT

- Standard deterministic protocol:

$$F_d^{\mathrm{std}} \geq 1 - \frac{d^2 - 1}{N}.$$

[Ishizaka and Hiroshima 2008; Beigi and König 2011]

- Converse bound for arbitrary deterministic protocols:

$$F_d^* \leq 1 - \frac{1}{4(d-1)N^2} + O(N^{-3}).$$

[Ishizaka 2015]

- Closed forms for $d = 2$: [Ishizaka and Hiroshima 2009]

$$F_2^{\mathrm{std}} = F_2^{\mathrm{EPR}} = 1 - \frac{3}{4N} + o(N^{-1})$$

$$p_2^{\mathrm{EPR}} \sim 1 - \left(\frac{8}{\pi N}\right)^{-1/2} + o(N^{-1/2}).$$

Existing results: optimal performance of PBT

- PBT has a lot of inherent **symmetries**

  $\longrightarrow$ use **representation theory** (RT)!

- Leads to **exact expressions** for F_d and p_d in terms of RT quantities. [Studziński et al. 2017] and [Mozrzymas et al. 2017]

- **Our main results:** Asymptotics of these expressions for F_d^*, F_d^{std} and p_d^{EPR} to first order.

- This talk focuses on
  - **standard deterministic protocol** F_d^{std};
  - **EPR probabilistic protocol** p_d^{EPR}.

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Natural symmetries of PBT

Notation: U_d … unitary group; S_N … symmetric group.

Permutation symmetry

Every port B_i is equally good for teleportation

$$\longrightarrow S_N\text{-symmetry of } \rho_{B^N} = \mathrm{Tr}_{A^N} \varphi_{A^N B^N}.$$

Same symmetry for POVM elements

$$\longrightarrow S_N\text{-action on } \{E^{(i)}\}_i.$$

Unitary invariance

The protocol works equally well for all input states

$$\longrightarrow U_d\text{-symmetry of } \rho_{B^N}.$$

Natural symmetries of PBT

Proposition: Symmetries of PBT

Every PBT protocol Λ can be symmetrized to a protocol Λ_s with $F(\Lambda_s, \mathrm{id}) \geq F(\Lambda, \mathrm{id})$, satisfying:

- Resource state $\varphi_{A^N B^N}$ is a purification of a symmetric Werner state, i.e., invariant under $U_A^{\otimes N} \otimes \overline{U}_B^{\otimes N}$ and S_N.
- S_N acts on $\{E^{(i)}\}_i$, and each $E^{(i)}$ is invariant under $\overline{U}_{A_0} \otimes U_A^{\otimes N}$.
- Λ_s is unitarily covariant.

”Folklore” results, proofs in C. Majenz’s PhD thesis and our paper.

Schur-Weyl duality

- Resource state $\varphi_{A^N B^N}$ invariant under action of U_d S_N
  $\longrightarrow$ structure determined by **Schur-Weyl duality**.
- Group actions:

$$|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \xmapsto{\pi \in S_N} |\psi_{\pi^{-1}(1)}\rangle \otimes \ldots \otimes |\psi_{\pi^{-1}(N)}\rangle$$

$$|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \xmapsto{U \in U_d} U|\psi_1\rangle \otimes \ldots \otimes U|\psi_N\rangle$$

- **Schur-Weyl decomposition:**

$$(\mathbb{C}^d)^{\otimes N} \cong \bigoplus_{\lambda \vdash_d N} [\lambda] \otimes V_\lambda$$

- $\lambda \vdash_d N$: Young diagram with N boxes and at most d rows.
- **Irreducible representations:**
  - $[\lambda]$ is an irrep of S_N with $\dim[\lambda] = d_\lambda$.
  - V_λ is an irrep of U_d with $\dim V_\lambda = m_{d,\lambda}$.

Exact expressions for F_d and p_d using RT

[Studziński et al. 2017; Mozrzymas et al. 2017]

Standard deterministic protocol

$$F_d^{std} = \frac{1}{d^{N-2}} \sum_{\alpha \vdash_d N-1} \left(\sum_{\mu=\alpha+\Box} \sqrt{d_\mu m_{d,\mu}} \right)^2,$$

where $\mu = \alpha + \Box$ denotes a Young diagram $\mu \vdash_d N$ obtained from $\alpha \vdash_d N-1$ by adding a single box (!).

EPR probabilistic protocol

$$p_d^{\mathrm{EPR}} = \frac{1}{d^N} \sum_{\alpha \vdash_d N-1} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}$$

where μ^* is the Young diagram obtained from $\alpha \vdash_d N-1$ by adding a single box such that $N\frac{m_{d,\mu} d_\alpha}{m_\alpha d_\mu}$ is maximal.

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Main results

Main result 1: standard deterministic PBT

For deterministic PBT using PGM and EPR pairs, we prove:

$$F_d^{\mathrm{std}} = 1 - \frac{d^2 - 1}{4N} + O(N^{-3/2+\delta}) \qquad \text{for any } \delta > 0.$$

- Recovers qubit result $F_2^{\mathrm{std}} = 1 - \frac{3}{4N} + o(N^{-1})$.
- Shows that $F_d^{\mathrm{std}} \geq 1 - \frac{d^2-1}{N}$ is not tight, confirming numerical evidence.

Main results

Main result 2: probabilistic PBT

For probabilistic PBT using EPR, we prove:

$$p_d^{\text{EPR}} = 1 - \sqrt{\frac{d}{N-1}}\mathbb{E}[\lambda_{\max}(\mathbf{G})] + o(N^{-1}),$$

where **G** is a Gaussian unitary, i.e, a Hermitian, traceless random $d \times d$ matrix with independent Gaussian RVs as entries.

- For qubits (i.e., $d = 2$ and **G** is a 2×2 matrix):

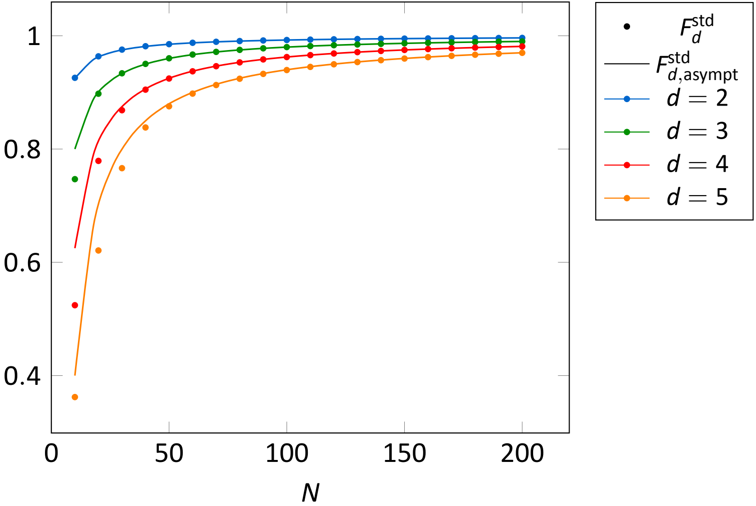
$$\mathbb{E}[\lambda_{\max}(\mathbf{G})] = 2\pi^{-1/2}.$$

- Hence, our result "corrects" the qubit result

$$p_2^{\text{EPR}} \sim 1 - \sqrt{\frac{8}{\pi N}} + o(N^{-1/2}).$$ [Ishizaka and Hiroshima 2009]

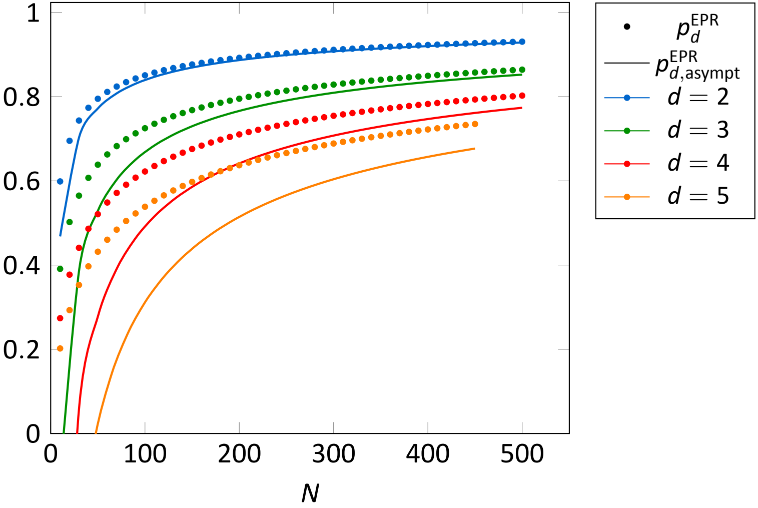
- Arbitrary d: use bounds on $\mathbb{E}[\lambda_{\max}(\mathbf{G})]$.

Standard deterministic protocol



$$F_d^{\mathrm{std}} = \frac{1}{d^{N-2}} \sum_{\alpha \vdash_d N-1} \left(\sum_{\mu = \alpha + \square} \sqrt{d_\mu m_{d,\mu}} \right)^2 \qquad F_{d,\mathrm{asympt}}^{\mathrm{std}} = 1 - \frac{d^2-1}{4N}$$

EPR probabilistic protocol



$$p_d^{\mathrm{EPR}} = \frac{1}{d^N} \sum_{\alpha \vdash_d N-1} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}$$

$$p_{d,\mathrm{asympt}}^{\mathrm{EPR}} = 1 - \mathbb{E}[\lambda_{\max}(\mathbf{G})] \sqrt{\frac{d}{N-1}}$$

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Spectrum estimation & random matrix theory

- Recall **Schur-Weyl duality:**

$$(\mathbb{C}^d)^{\otimes N} \cong \bigoplus_{\lambda \vdash_d N} [\lambda] \otimes V_\lambda.$$

- Consider the projective measurement $\{P_\lambda\}_{\lambda \vdash_d N}$, where P_λ is the orthogonal projection onto $[\lambda] \otimes V_\lambda$.
- Intuition: $\{P_\lambda\}_{\lambda \vdash_d N}$ respects the U_d- and S_N-symmetries of the spectrum estimation problem.

Spectrum estimation [Keyl and Werner 2001]

Let $\mathbf{Y}_N$ denote the outcome of the measurement $\{P_\lambda\}_{\lambda \vdash_d N}$ applied to $\rho^{\otimes N}$ where ρ is a state. Then, as $N \to \infty$,

$$\frac{1}{N}\mathbf{Y}_N \xrightarrow{D} \operatorname{spec}(\rho).$$

Spectrum estimation & random matrix theory

- For the completely mixed state $\tau = \frac{1}{d}\mathbb{1}$, the corresponding probability distribution is called **Schur-Weyl distribution**:

$$p_{d,N}(\lambda) = \mathrm{Tr}(P_\lambda \tau^{\otimes N}) = \frac{1}{d^N} d_\lambda m_{d,\lambda}.$$

- Spectrum estimation: For the RV $\mathbf{Y}_N^\tau$ obtained from applying the measurement $\{P_\lambda\}_{\lambda \vdash_d N}$ to τ, we have

$$\tfrac{1}{N}\mathbf{Y}_N^\tau \xrightarrow{D} (1/d, \ldots, 1/d).$$

- What about a "central limit theorem" version of this describing fluctuations of Young diagrams?

Spectrum estimation & random matrix theory

- To make this exact, define the centered and normalized RV

$$\mathbf{A}_N = \frac{\boldsymbol{\lambda}_N - (N/d, \ldots, N/d)}{\sqrt{N/d}}$$

where $\boldsymbol{\lambda}_N \sim p_{d,N}$ takes values in Young diagrams $\{\lambda \vdash_d N\}$.

- Let $\mathbf{M}$ be the **Gaussian unitary ensemble** $\mathrm{GUE}(d)$: a Hermitian random matrix whose entries are independent Gaussian RVs.

  (df: $\exp(-\frac{1}{2}\operatorname{Tr}\mathbf{H}^2)$ where $\mathbf{H}$ is a Hermitian matrix-valued RV.)

- Define $\mathbf{M}_0 = \mathbf{M} - \frac{\operatorname{Tr}(\mathbf{M})}{d}\mathbb{1}$, called the **traceless Gaussian unitary ensemble** $\mathrm{GUE}_0(d)$.

Main technical result

Fluctuations of Schur-Weyl distribution [Johansson 2001]

For the RV $\mathbf{A}_N = \sqrt{\frac{d}{N}}(\boldsymbol{\lambda}_N - (N/d, \ldots, N/d))$,

$$\mathbf{A}_N \xrightarrow{D} \mathrm{spec}(\mathbf{G}),$$

where $\mathbf{G} \sim \mathrm{GUE}_0(d)$.

Note that $\mathrm{spec}(\mathbf{G}) \xrightarrow{d\to\infty}_{\text{a.s.}}$ Wigner's semicircle law.

Main technical result (informal)

Strengthening of Johansson's result:

$$\mathbb{E}[g(\mathbf{A}_N)] \xrightarrow{n\to\infty} \mathbb{E}[g(\mathrm{spec}(\mathbf{G}))]$$

for "suitable" functions g.

Application of Johansson strengthening

- **Proof idea of asymptotics for standard and EPR protocol:** apply convergence of expectation values to exact RT formulas by rewriting them as expectation values over Schur-Weyl distribution.
- **Main principle:** Computing expectation values of (functions of) GUE-distributed matrices is much easier!
- Example: probabilistic PBT
$$p_d^{\mathrm{EPR}} = \frac{1}{d^N} \sum_{\alpha \vdash_d N-1} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}$$
- Rewrite $p_d^{\mathrm{EPR}} = \frac{N}{d} \mathbb{E}_\alpha[(\alpha_1 + d)^{-1}]$ and apply technical result.

More results in the paper

Main result 3: fully optimized deterministic PBT

- Achievability bound:

$$F_d^* \geq 1 - \frac{d^5 + O(d^{9/2})}{4\sqrt{2}N^2} + O(N^{-3}).$$

- Converse bound:

$$F_d^* \leq 1 - \frac{d^2 - 1}{16N^2}.$$

- **Asymptotics of optimal deterministic PBT** are given by

$$F_d^* = 1 - \Theta(N^{-2}).$$

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Summary

- We discussed two variants of port-based teleportation (PBT):
  - deterministic PBT with entanglement fidelity F_d;
  - probabilistic PBT with success probability p_d.
- **Inherent symmetries:** closed representation-theoretic formulas for F_d and p_d. [Studziński et al. 2017; Mozrzymas et al. 2017]
- **Standard protocols:** use connection between Young diagrams and GUE to determine asymptotics.
- We also determine asymptotics of fully optimized case using different proof technique.

Open problems

Connection between Schur-Weyl distribution and GUE seems very fruitful: get asymptotics for other ”symmetric” tasks?

For the optimal deterministic case, achievability bound is optimal in N, but not in d-dependence: improvement?

We considered natural limit of fixed d and $N \to \infty$.
What about the limit $N, d \to \infty$ with $N/d^2 = \text{const}$?

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Thank you very much for your attention!