Convex integration and compressible Euler system

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(Full) Euler system in conservative variables

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

Energy balance

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Constitutive relations

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e, \ (\gamma - 1)\varrho e = p, \ \gamma > 1$$

Second law – entropy

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \ S = \varrho s$$

Entropy balance

$$\partial_t S + \mathrm{div}_x(s\mathbf{m}) = 0, \ \partial_t S + \mathrm{div}_x(s\mathbf{m}) \boxed{\geq} 0$$

Boyle–Mariot law

$$p = \varrho\vartheta, \ e = c_v\vartheta, c_v = \frac{1}{\gamma - 1}, \ s = c_v \log(\vartheta) - \log(\varrho)$$

Renormalized entropy balance

$$\partial_t s + \frac{\mathbf{m}}{\varrho} \cdot \nabla_x s = (\geq)0, \ \partial_t G(s) + \frac{\mathbf{m}}{\varrho} \cdot \nabla_x G(s) = (\geq)0, \ G' \geq 0$$

Isentropic (barotropic) Euler system

Constant entropy

$$s = \bar{s},\ \ p = \varrho\vartheta = \exp\left(\frac{\bar{s}}{c_v}\right)\varrho^{\gamma},\ \ p = p(\varrho),\ \ p' \geq 0$$

Total energy

$$E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

Total energy balance

$$\partial_t E + \mathrm{div}_x \left[(E + p(\varrho)) \frac{\mathbf{m}}{\varrho} \right] = (\leq) 0$$

Energetically closed system

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} E \ \mathrm{d}x = (\leq) 0$$

Data

Initial data

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Impermeable boundary

$$\mathbf{m}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

First ansatz – constant thermostatic variables

Constant density, temperature (internal energy), and total energy

$$\varrho = \varrho_\Omega > 0,\ \vartheta = \vartheta_\Omega > 0 \ \Rightarrow \ S = S_\Omega > 0$$

$$E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho_\Omega} + \varrho_\Omega e(\varrho_\Omega, \vartheta_\Omega) \boxed{= E_\Omega}$$

Mass and entropy conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0,\ \partial_t S + \mathrm{div}_x (s\mathbf{m}) = 0 \ \Rightarrow \ \boxed{\mathrm{div}_x \mathbf{m} = 0}$$

Total energy balance

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0 \ \Rightarrow \ \boxed{\mathrm{div}_x \mathbf{m} = 0}$$

Momentum balance

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho_\Omega} \right) = 0$$

Incompressible Euler system with constant pressure

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho_\Omega} - \frac{1}{N} \frac{|\mathbf{m}|^2}{\varrho_\Omega} \mathbb{I} \right) = 0, \ \mathrm{div}_x \mathbf{m} = 0$$

Prescribed (constant) kinetic energy energy

$$\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho_\Omega} = -\frac{N}{2} p(\varrho_\Omega, \vartheta_\Omega) + \boxed{\Lambda(t)}$$

Weak formulation – no flux boundary conditions

$$\left[\int_\Omega \mathbf{m} \cdot \phi \, \mathrm{d}\mathbf{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho_\Omega} - \frac{1}{N} \frac{|\mathbf{m}|^2}{\varrho_\Omega} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = 0$$

$$\int_0^T \int_\Omega \mathbf{m} \cdot \nabla_x \varphi \, \mathrm{d}x\mathrm{d}t = 0$$

$$\boldsymbol{\varphi} \in C^1([0, T] \times \overline{\Omega}; R^N), \ \varphi \in C^1([0, T] \times \overline{\Omega})$$

Convex integration [DeLellis and Székelyhidi]

Relaxation – subsolutions

$$\partial_t \mathbf{m} + \mathrm{div}_x \mathbb{V} = 0,\ \mathrm{div}_x \mathbf{m} = 0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0,\ \mathbb{V} \in C^1([0,T] \times \overline{\Omega}; R^{N\times N}_{\mathrm{sym},0})$$

Convex constraint

$$\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho_\Omega} \leq \frac{N}{2}\lambda_{\max}\left[\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho_\Omega} - \mathbb{V}\right] < -\frac{N}{2}p(\varrho_\Omega, \vartheta_\Omega) + \Lambda(t) \equiv \overline{E}$$

Algebraic relations

$$(\mathbf{v}, \mathbb{V}) \mapsto \lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{V}\right] \text{ convex}$$

$$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{V}\right]$$

$$\frac{1}{2}|\mathbf{v}|^2 = \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{V}\right] \Rightarrow \mathbb{V} = \mathbf{v}\otimes\mathbf{v} - \frac{1}{N}|\mathbf{v}|^2\mathbb{I}$$

Convex integration [DeLellis and Székelyhidi]

Non–empty set of subsolutions

$$\overline{E} \text{ large enough } \Rightarrow \text{ set of subsolutions is non–empty}$$

$$\overline{E} - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho_\Omega} > 0$$

Energy defect functional

$$I[\mathbf{v}] = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho_\Omega} - \overline{E} < 0 \text{ convex}$$

$I[\mathbf{m}] = 0 \Rightarrow$ **m** is a (weak) solution of the constant pressure Euler system

$$\mathbf{m} \text{ is a point of continuity of } I \Rightarrow I[\mathbf{m}] = 0$$

Convex integration [DeLellis and Székelyhidi]

Oscillatory lemma

Let

$$\mathbf{m} \text{ with the associated flux } \mathbb{V}$$

be a subsolution.
Then there exists a sequence $\{\mathbf{v}_n, \mathbb{U}_n\}_{n=1}^{\infty}$ such that

- $$\mathbf{v}_n \in C_c^{\infty}((0,T)\times\Omega; R^N),\ \mathbb{U}_n \in C_c^{\infty}((0,T)\times\Omega; R^{N\times N}_{\mathrm{sym},0})$$
- $\mathbf{m} + \mathbf{v}_n$ with the associated flux $\mathbb{V} + \mathbb{U}_n$ are subsolutions
- $$\mathbf{v}_n \to 0 \text{ weakly in } L^2((0,T)\times\Omega)$$
- $$\liminf_{n\to\infty} \int_0^T \int_\Omega \frac{1}{2}\frac{|\mathbf{v}_n|^2}{\varrho\Omega}\ \mathrm{d}x\mathrm{d}t \geq c(N,\overline{E}) \int_0^T \int_\Omega \left(\overline{E} - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho\Omega}\right)^2\ \mathrm{d}x\mathrm{d}t$$

Conclusion for the Euler system with constant (zero) pressure

Conclusion A

Given $\varrho_\Omega > 0$, $\mathbf{m}_0 \in C^1(\overline{\Omega}; R^N)$, $\mathrm{div}_x \mathbf{m}_0 = 0$, $\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$, there exist (infinitely many) $\overline{E} > 0$ such that the Euler system with constant pressure admits infinitely many weak solutions. The solution may experience the initial energy "jump".

Conclusion B

Given $\varrho_\Omega > 0$, there exist (infinitely many) $\mathbf{m}_0$, $\overline{E} > 0$ such that the Euler system with constant pressure admits infinitely many weak solutions with the energy continuous at $t = 0$

Application to the full Euler system

Full Euler system with piece–wise constant data

Suppose that $\Omega \subset R^N$ is a bounded domain,

$$\Omega = \cup_{i>0}\overline{\Omega}_i,\ \ \Omega_i \text{ domains } \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j.$$

Let the initial data ϱ_0, ϑ_0 be given,

$$\varrho_0|_{\Omega_i} = \varrho_{\Omega_i} > 0,\ \ \vartheta_0|_{\Omega_i} = \vartheta_{\Omega_i} > 0,\ \ i = 1, 2, \dots$$

The there exist infinitely many $\mathbf{m}_0 \in L^\infty(\Omega; R^N)$ such that the full Euler system supplemented with the impermeability boundary condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

admits infinitely many weak solutions satisfying the entropy (in)equality.

Application to stochastically driven Euler system

Euler system with Stratonowich intergral

$$\mathrm{d}\varrho + \mathrm{div}_x \mathbf{m} \mathrm{d}t = 0$$

$$\mathrm{d}\mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) \mathrm{d}t + \nabla_x p \mathrm{d}t = -\frac{1}{2} \mathbf{m} \circ \mathrm{d}W$$

$$\mathrm{d}E + \mathrm{div}_x \left((E + p) \frac{\mathbf{m}}{\varrho} \right) \mathrm{d}t = -E \circ \mathrm{d}W,$$

Entropy inequality

$$\mathrm{d}(\varrho s) + \mathrm{div}_x (s\mathbf{m}) \mathrm{d}t \geq -c_v \varrho \circ \mathrm{d}W.$$

Impermeability condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Second ansatz – barotropic Euler system

Equation of continuity

$$\mathbf{m} = \mathbf{v} + \nabla_x \Phi, \ \mathrm{div}_x \mathbf{v} = 0, \ \Delta_x \Phi = \mathrm{div}_x \mathbf{m}, \ (\nabla_x \Phi - \mathbf{m}) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t \varrho + \Delta_x \Phi = 0$$

Momentum equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} + \partial_t \Phi \mathbb{I} + p(\varrho)\mathbb{I} \right) = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \mathbb{I} \right) = 0$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} = -\frac{N}{2} \left(p(\varrho) + \partial_t \Phi \right) + \Lambda(t)$$

Ill posedness for barotropic Euler system

Theorem

Let $\Omega \subset R^N$ be a bounded domain, $N = 2, 3$. Let $\mathcal{R}$ denotes the set of all functions bounded and continuous in $\overline{\Omega}$ with the exception of a set of Lebesgue measure zero. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times. Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^q(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^q(\Omega; R^N)),\ q > 1,$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i,\ i = 1, 2, \dots .$$

General "Euler" system

Incompressibility constraint, initial data

$$\mathrm{div}_x \mathbf{v} = 0, \ \ \mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Momentum equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{b}[\mathbf{v}]) \otimes (\mathbf{v} + \mathbf{b}[\mathbf{v}])}{r[\mathbf{v}]} - \frac{1}{N} \frac{|\mathbf{v} + \mathbf{b}[\mathbf{v}]|^2}{r[\mathbf{b}]} \mathbb{I} + \mathbb{M}[\mathbf{v}] \right) = 0$$

Total energy

$$\frac{1}{2} \frac{|\mathbf{v} + b[\mathbf{v}]|^2}{r[\mathbf{v}]} = E[\mathbf{v}]$$

Abstract operators

$$\mathbf{v} \mapsto [\mathbf{b}, r, \mathbb{M}, E][\mathbf{v}]$$

continuous from $L^\infty_{\mathrm{weak}-(*)}$ to BC (bounded continuous)

Conclusion for the general "Euler" system

Result A

$\Omega \subset R^N$ a bounded domain, $N = 2, 3$. If the set of subsolutions is non–empty, there exists infinitely many weak solutions. They may experience the initial energy jump.

Result B

$\Omega \subset R^N$ a bounded domain, $N = 2, 3$. If the set of subsolutions is non–empty, there exist infinitely may initial data $\mathbf{v}_0 \in L^\infty(\Omega; R^N)$ such that the problem admits infinitely many weak solutions with the total energy continous at $t = 0$.

Example: Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\mathbf{u}$$

Third ansatz – 1D Riemann problem

Riemann problem in 1D

$$\partial_t \varrho + \partial_{x_1}(\varrho u) = 0, \;\; \partial_t(\varrho u) + \partial_{x_1}(\varrho u^2) + \partial_{x_1}\varrho^\gamma = 0$$

$$\varrho_0 = \varrho(x_1) = \begin{cases} \varrho_L \text{ for } x_1 < 0, \\ \\ \varrho_R \text{ for } x_1 \geq 0 \end{cases}$$

$$\varrho(0,\cdot) = \varrho_0, \;\; u(0,\cdot) = u_0$$

$$u_0 = u(x_1) = \begin{cases} u_L \text{ for } x_1 < 0, \\ \\ u_R \text{ for } x_1 \geq 0 \end{cases}$$

Extension to multi-D

$$\varrho(x_1,\cdot) = \varrho(x_1), \;\; \mathbf{u}(x_1,\cdot) = [u(x_1), 0, \ldots, 0]$$

+periodic boundary conditions

Results for the Riemann data - [Chiodaroli, DeLellis, Kreml, Markfelder,...]

A "generic" result for shocks

The extended problem in 2 and 3D admits infinitely many weak solutions satisfying the energy inequality whenever the 1D data give rise to a shock wave

Corollary

Given $T > 0$, there exist Lipschitz initial data such that the isentropic Euler system admits infinitely many admissible weak solutions in $(0, T)$. Similar results hold also for the complete Euler system

Smooth initial data [Chiodaroli, Kreml, Mácha, Schwarzacher]

There exist smooth initial data and $T > 0$ such that the isentropic Euler system admits infinitely many admissible weak solutions in $(0, T)$.