Onsager's conjecture for general conservation laws

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Introduction: the principle of conservation of energy for classical solutions

Let us first focus our attention on the incompressible Euler system

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0, \\ \operatorname{div} u = 0, \end{aligned}$$

If u is a classical solution, then multiplying the balance equation by u we obtain

$$\frac{1}{2}\partial_t |u|^2 + \operatorname{div}\left(\frac{1}{2}|u|^2 u\right) + u \cdot \nabla p = 0.$$

Integrating the last equality over the space domain Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|u(x,t)|^2 \mathrm{d}x = \int_{\partial\Omega} \left(\frac{1}{2}|u|^2 u\right) \cdot n ds.$$

Integrating over time in $(0,t)$ (with $u \cdot n = 0$ on $\partial\Omega$), gives

$$\int_\Omega \frac{1}{2}|u(x,t)|^2 \, \mathrm{d}x = \int_\Omega \frac{1}{2}|u(x,0)|^2 \, \mathrm{d}x.$$

Weak solutions

However, if u is a weak solution, then

$$\int_\Omega \frac{1}{2}|u(x,t)|^2 \, \mathrm{d}x = \int_\Omega \frac{1}{2}|u(x,0)|^2 \, \mathrm{d}x.$$

might not hold. Technically, the problem is that u might not be regular enough to justify the chain rule in the above derivation. Motivated by the laws of turbulence Onsager postulated that there is a critical regularity for a weak solution to be a conservative one:

Conjecture, 1949

Let u be a weak solution of incompressible Euler system

- If $u \in \mathrm{C}^\alpha$ with $\alpha > \frac{1}{3}$, then the energy is conserved.
- For any $\alpha < \frac{1}{3}$ there exists a weak solution $u \in C^\alpha$ which does not conserve the energy.

Onsager conjecture for incompressible Euler system

Weak solutions of the incompressible Euler equations which do not conserve energy were constructed:

- Scheffer '93, Shnirelman '97 constructed examples of weak solutions in $L^2(\mathbb{R}^2 \times \mathbb{R})$ compactly supported in space and time
- De Lellis and Székelyhidi showed how to construct weak solutions for given energy profile

Still incompressible case

- Significant progress has recently been made in constructing energy-dissipating solutions slightly below the Onsager regularity , see e.g.:

  - T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi, Anomalous dissipation for 1/5-Hölder Euler flows. Ann. of Math. (2), 2015
  - T. Buckmaster, C. De Lellis, and L. Székelyhidi, Dissipative Euler flows with Onsager-critical spatial regularity. Comm. Pure and Appl. Math., 2015.

  And the story is closed by the results:

  - **Philip Isett, A Proof of Onsager's Conjecture, Ann. of Math. 2018**
  - **Tristan Buckmaster, Camillo De Lellis, László Székelyhidi Jr., Vlad Vicol, Onsager's conjecture for admissible weak solutions, Comm. Pure Appl. Math. 2019**

Still incompressible case

Onsager conjecture:

If weak solution v has C^α (for $\alpha > \frac{1}{3}$) regularity then it conserves energy. In the opposite case it may not conserve energy.

The first part of this assertion was proved in

- P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.*, 1994.
- G. L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. *Phys. D*, 1994.
- A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity*, 2008.

The standard technique is based either on the convolution of the Euler system with a standard family of mollifiers or truncation in Fourier space based on Littlewood-Paley decomposition.

Besov spaces

The elements of Besov space $B_p^{\alpha,\infty}(\Omega)$, where $\Omega = (0, T) \times \mathbb{T}^d$ or $\Omega = \mathbb{T}^d$ are functions w for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{\xi \in \Omega} \frac{\|w(\cdot + \xi) - w\|_{L^p(\Omega \cap (\Omega - \xi))}}{|\xi|^\alpha}$$

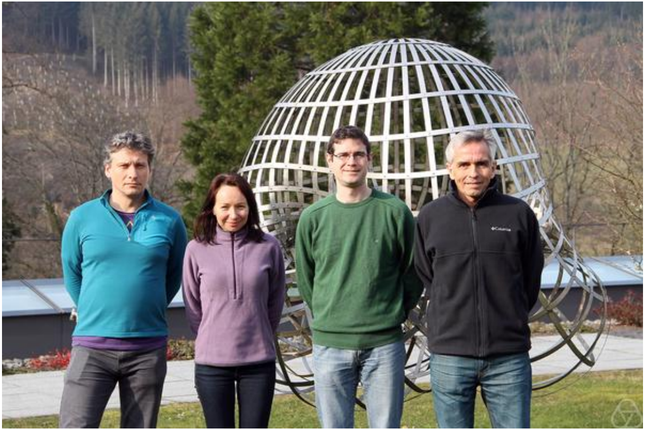
is finite (here $\Omega - \xi = \{x - \xi : x \in \Omega\}$).
It is then easy to check that the definition of the Besov spaces implies

$$\|w^\epsilon - w\|_{L^p(\Omega)} \leq C\epsilon^\alpha \|w\|_{B_p^{\alpha,\infty}(\Omega)}$$

and

$$\|\nabla w^\epsilon\|_{L^p(\Omega)} \leq C\epsilon^{\alpha-1} \|w\|_{B_p^{\alpha,\infty}(\Omega)}.$$

Onsager's conjecture for compressible Euler system



Compressible Euler system

We consider now the isentropic Euler equations,

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= 0, \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0. \end{aligned} \tag{1}$$

We will use the notation for the so-called pressure potential defined as

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} dr.$$

E. Feireisl, P. G., A. Świerczewska-Gwiazda, and E. Wiedemann.
Regularity and Energy Conservation for the Compressible Euler Equations.
Arch. Rational Mech. Anal., 2017.

Theorem

Let ϱ, u *be a solution of* (1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d),\ \ \varrho, \varrho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), 0 \le \underline{\varrho} \le \varrho \le \overline{\varrho}$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$, *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\beta > \max\left\{1 - 2\alpha; \frac{1-\alpha}{2}\right\}. \tag{2}$$

Assume further that $p \in C^2[\underline{\varrho}, \overline{\varrho}]$, *and, in addition*

$$p'(0) = 0 \text{ as soon as } \underline{\varrho} = 0.$$

Then the energy is locally conserved in the sense of distributions on $(0,T)\times\Omega$, *i.e.*

$$\partial_t\left(\frac{1}{2}\varrho|u|^2 + P(\varrho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|u|^2 + p(\varrho) + P(\varrho)\right)u\right] = 0.$$

The Divergence-Measure Condition

Asssume the hypotheses of the FGŚW Theorem, except we now allow for $1 < \gamma < 2$ and $\rho \geq 0$. Assume in addition div *v* is a bounded measure. Then the energy is conserved.

I. Akramov, T. Dębiec., J. Skipper, E. Wiedemann. Energy conservation for the compressible Euler and Navier-Stokes equations in vacuum. *to appear in Analysis & PDE*, 2019.

Sharpness of assumptions

Shocks provide examples that show that our assumptions are sharp:

- A shock solution dissipates energy, but ρ and u are in $BV \cap L^\infty$, which embeds into $B_3^{1/3,\infty}$.
- Hence such a solution satisfies (2) with equality but fails to satisfy the conclusion.

Time regularity

The hypothesis on temporal regularity can be relaxed provided

$$\underline{\varrho} > 0$$

Indeed, in this case $\frac{(\varrho u)^\epsilon}{\varrho^\epsilon}$ can be used as a test function in the momentum equation, cf.

T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations. *J. Differential Equations*, 2016.

Some references to other systems

- R. E. Caflisch, I. Klapper, and G. Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 1997.
- E. Kang and J. Lee. Remarks on the magnetic helicity and energy conservation for ideal magneto-hydrodynamics. *Nonlinearity*, 2007.
- R. Shvydkoy. On the energy of inviscid singular flows. *J. Math. Anal. Appl.*, 2009.
- C. Yu. Energy conservation for the weak solutions of the compressible Navier–Stokes equations. *Arch. Rational Mech. Anal.*, 2017.
- T. D. Drivas and G. L. Eyink. An Onsager singularity theorem for turbulent solutions of compressible Euler equations. *Comm. in Math. Physics*, 2017.

General conservation laws

- It is easy to notice similarities in the statements regarding sufficient regularity conditions guaranteeing energy/entropy conservation for various systems of equations of fluid dynamics.
- Especially the differentiability exponent of $\frac{1}{3}$ is a recurring condition.
- One might therefore anticipate that a general statement could be made, which would cover all the above examples and more. Indeed, consider a general conservation law of the form

$$\mathrm{div}_X(G(U(X))) = 0.$$

P. G., M. Michálek, A. Świerczewska-Gwiazda A note on weak solutions of conservation laws and energy/entropy conservation. *Arch. Rational Mech. Anal.*, 2018.

We consider the conservation law of the form

$$\mathrm{div}_X(G(U(X))) = 0. \tag{3}$$

Here $U : \mathcal{X} \to \mathcal{O}$ is an unknown and $G : \mathcal{O} \to \mathbb{M}^{n\times(d+1)}$ is a given, where $\mathcal{X}$ is an open subset of $\mathbb{R}^{d+1}$ or $\mathbb{T}^3 \times \mathbb{R}$ and the set $\mathcal{O}$ is open in $\mathbb{R}^n$. It is easy to see that any classical solution to (3) satisfies also

$$\mathrm{div}_X(Q(U(X))) = 0, \tag{4}$$

where $Q : \mathcal{O} \to \mathbb{R}^{s\times(d+1)}$ is a smooth function such that

$$D_U Q_j(U) = \mathfrak{B}(U) D_U G_j(U), \quad \text{for all } U \in \mathcal{O},\ j \in 0, \cdots, k, \tag{5}$$

for some smooth function $\mathfrak{B} : \mathcal{O} \to \mathbb{M}^{s\times n}$. The function Q is called a *companion* of G and equation (4) is called a *companion law* of the conservation law (3).

Weak solutions

In many applications some relevant companion laws are *conservation of energy* or *conservation of entropy*. We consider the standard definition of weak solutions to a conservation law.

Definition

We call the function U a weak solution to (3) if $G(U)$ is locally integrable in $\mathcal{X}$ and the equality

$$\int_{\mathcal{X}} G(U(X))\colon D_X\psi(X)dX = 0$$

holds for all smooth test functions $\psi\colon \mathcal{X} \to \mathbb{R}^n$ with a compact support in $\mathcal{X}$.

Does such an abstract framework cover any physical systems?

Consider the case where $\mathcal{X} = \mathbb{T}^3 \times (0, T)$ and we write $X = (x, t)$. Then G can be written in the form

$$G(U) = (F(U), A(U))$$

for some $A : \mathcal{O} \to \mathbb{R}^n$ and $F : \mathcal{O} \to \mathbb{R}^{n \times k}$, so that the conservation law (3) reads

$$\partial_t [A(U(x,t))] + \operatorname{div}_x F(U(x,t)) = 0,$$

or, in weak formulation,

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \psi(x,t) \cdot A(U(x,t)) + \nabla_x \psi(x,t) : F(U(x,t)) \, dxdt = 0$$

for any $\psi \in C_c^1(\mathbb{T}^3 \times (0,T); \mathbb{R}^n)$.

Does such an abstract framework covers any physical systems?

Setting $Q(U) = (q(U), \eta(U))$ for $q : \mathcal{O} \to \mathbb{R}^{s\times k}$ and $\eta : \mathcal{O} \to \mathbb{R}^s$, we accordingly consider companion laws of the form

$$\partial_t[\eta(U(x,t))] + \operatorname{div}_x q(U(x,t)) = 0,$$

where η and q satisfy

$$\begin{aligned} D_U\eta(U) &= B(U) D_U A(U), \\ D_U q_j(U) &= B(U) D_U F_j(U) \quad \text{for } j = 1, \dots, k \end{aligned}$$

for some smooth map $B : \mathcal{O} \to \mathbb{R}^{s\times n}$.

Example – Inviscid magnetohydrodynamics – compressible and incompressible

We will recall only the incompressible case. Let us consider the system

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v - h \otimes h) + \nabla_x (p + \frac{1}{2}|h|^2) &= 0, \\ \partial_t h + \operatorname{div}(v \otimes h - h \otimes v) &= 0, \\ \operatorname{div} v = 0, \qquad \operatorname{div} h &= 0 \end{aligned}$$

where $v \colon Q \to \mathbb{R}^n$ and $h \colon Q \to \mathbb{R}^n$ and $p \colon Q \to \mathbb{R}$. The system describes the motion of an ideal electrically conducting fluid. Here $U = (v, h, p)$, $A(U) = (v, h, 0)$, and

$$F(v, h) = \left(v \otimes v - h \otimes h + (p + \frac{1}{2}|h|^2)\mathbb{I}, v \otimes h - h \otimes v, v \right).$$

The entropy is given by $\eta = \frac{1}{2}(|v|^2 + |h|^2)$ and the entropy flux is $q = \frac{1}{2}(|v|^2 + |h|^2)v - (v \cdot h)h$.

Example – nonlinear elastodynamics

We recall a quasi-linear wave equation that might be interpreted as a model of nonlinear elastodynamics, when we understand $y\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ as a displacement vector

$$\frac{\partial^2 y}{\partial t^2} = \operatorname{div}_x S(\nabla y).$$

In the above equation S is a gradient of some function $G : \mathbb{M}^{3\times 3} \to [0,\infty)$. We rewrite this equation as a system, introducing the notation $v_i = \partial_t y_i$ and $\mathbb{F}_{i\alpha} = \frac{\partial y_i}{\partial x_\alpha}$. Then $U = (v, \mathbb{F})$ solves the system

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial G}{\partial \mathbb{F}_{i\alpha}}\right),$$
$$\frac{\partial \mathbb{F}_{i\alpha}}{\partial t} = \frac{\partial v_i}{\partial x_\alpha}.$$

With $A(U) \equiv id$ and $F(U) = \left(\frac{\partial G}{\partial \mathbb{F}_{i\alpha}}, v\right)$ we have an entropy $\eta(U) = \frac{1}{2}|v|^2 + G(\mathbb{F})$ and an entropy flux $q_\alpha(U) = v_i \frac{\partial G(\mathbb{F})}{\partial \mathbb{F}_{i\alpha}}$.

How much regularity of a weak solution is required so that it also satisfies the companion law?

Theorem (P.G., Michálek, Świerczewska-Gwiazda, ARMA 2018)

Let $U \in B^{\alpha}_{3,\infty}(\mathcal{X};\mathcal{O})$ be a weak solution of (3) with $\alpha > \frac{1}{3}$. Assume that $G \in C^2$ is endowed with a companion law with flux $Q \in C)$ for which there exists $\mathcal{B} \in C^1$ related through identity (5) and all the following conditions hold

$$\begin{aligned} \mathcal{O} \text{ is convex}, \\ \mathcal{B} \in W^{1,\infty}(\mathcal{O};\mathbb{M}^{1\times n}), \\ |Q(V)| \le C(1+|V|^3) \text{ for all } V \in \mathcal{O}, \\ \sup_{i,j\in 1,\dots,d} \|\partial_{U_i}\partial_{U_j} G(U)\|_{C(\mathcal{O};\mathbb{M}^{n\times(k+1)})} < +\infty. \end{aligned}$$

Then U is a weak solution of the companion law (4) with the flux Q.

The essential part of the proof of this Theorem pertains the estimation of the nonlinear commutator

$$[G(U)]_\varepsilon - G([U]_\varepsilon).$$

It is based on the following observation:

Lemma

Let $\mathcal{O}$ be a convex set, $U \in L^2_{loc}(\mathcal{X}, \mathcal{O})$, $G \in C^2(\mathcal{O}; \mathbb{R}^n)$ and let

$$\sup_{i,j \in 1,\ldots,d} \|\partial_{U_i} \partial_{U_j} G(U)\|_{L^\infty(\mathcal{O})} < +\infty.$$

Then there exists $C > 0$ depending only on η_1, second derivatives of G and k (dimension of $\mathcal{O}$) such that

$$\begin{aligned} &\|[G(U)]_\varepsilon - G([U]_\varepsilon)\|_{L^q(K)} \\ &\leq C\Big(\|[U]_\varepsilon - U\|^2_{L^{2q}(K)} + \sup_{Y \in \mathrm{supp}\ \eta_\varepsilon} \|U(\cdot) - U(\cdot - Y)\|^2_{L^{2q}(K)}\Big) \end{aligned}$$

for $q \in [1, \infty)$, where $K \subseteq \mathcal{X}$ satisfies $K^\varepsilon \subseteq \mathcal{X}$.

Remarks

- Due to the assumption on the convexity of $\mathcal{O}$ the previous theorem could be deduced from the result for compressible Euler system (Feireisl, G., Świerczewska-Gwiazda, Wiedemann ARMA 2017).
- It is worth noting that the convexity of $\mathcal{O}$ might not be natural for all applications (this is e.g. the case of the polyconvex elasticity).

A few words about polyconvex elasticity

Let us consider the evolution equations of nonlinear elasticity

$$\left.\begin{aligned} \partial_t F &= \nabla_x \mathbf{v} \\ \partial_t \mathbf{v} &= \mathrm{div}_x \left(D_F W(F) \right) \end{aligned}\right. \quad \text{in } \mathcal{X},$$

for an unknown matrix field $F\colon \mathcal{X} \to \mathbb{M}^{k\times k}$, and an unknown vector field $\mathbf{v}\colon \mathcal{X} \to \mathbb{R}^k$. Function $W\colon \mathcal{U} \to \mathbb{R}$ is given. For many applications, $\mathcal{U} = \mathbb{M}^{k\times k}_+$ where $\mathbb{M}^{k\times k}_+$ denotes the subset of $\mathbb{M}^{k\times k}$ containing only matrices having positive determinant. Let us point out that $\mathbb{M}^{k\times k}_+$ is a non–convex connected set.

To this purpose, we study the case of non–convex $\mathcal{O}$

- Having $\mathcal{O}$ non–convex, we face the problem that $[U]_\varepsilon$ does not have to belong to $\mathcal{O}$.
- The convexity was crucial to conduct the Taylor expansion argument in error estimates.
- However, a suitable extension of functions G, $\mathcal{B}$ and Q does not alter the previous proof significantly.

How much regularity of a weak solution is required so that it also satisfies the companion law?

Theorem

Let $U \in B_3^{\alpha,\infty}(\mathcal{X};\mathcal{O})$ be a weak solution of (3) with $\alpha > \frac{1}{3}$. Assume that $G \in \mathrm{C}^2(\mathcal{O};\mathbb{M}^{n\times(d+1)})$ is endowed with a companion law with flux $Q \in \mathrm{C}(\mathcal{O};\mathbb{M}^{s\times(d+1)})$ for which there exists $\mathfrak{B} \in \mathrm{C}^1(\mathcal{O};\mathbb{M}^{s\times n})$ related through identity (5) and the essential image of U is compact in $\mathcal{O}$.
Then U is a weak solution of the companion law (4) with the flux Q.

Remarks

- the generality of the above theorem is achieved at the expense of optimality of the assumptions.
- However given additional information on the structure of the problem at hand one might be able to relax some of these assumptions.
- the theorem provides for instance a conservation of energy result for the system of polyconvex elastodynamics, compressible hydrodynamics et al.

Result of Constantin, E, Titi

Theorem

Let $u \in L^3([0,T], B_3^{\alpha,\infty}(\mathbb{T}^3)) \cap \mathrm{C}([0,T], L^2(\mathbb{T}^3))$ *be a weak solution of the incompressible Euler system. If* $\alpha > \frac{1}{3}$*, then*

$$\int_{\mathbb{T}^3} \frac{1}{2}|u(x,t)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^3} \frac{1}{2}|u(x,0)|^2 \,\mathrm{d}x$$

for each $t \in [0,T]$.

Additional structure of equations

The first lemma gives a sufficient condition to drop the Besov regularity with respect to some variables. It is connected with the columns of G.

Lemma

Let $G = (G_1, \dots, G_s, G_{s+1}, \dots G_k)$ where $G_1, \dots, G_s$ are affine vector–valued functions and $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ where $\mathcal{Y} \subseteq \mathbb{R}^s$ and $\mathcal{Z} \subseteq \mathbb{R}^{k+1-s}$. Then it is enough to assume that $U \in L^3(\mathcal{Y}; B^{\alpha}_{3,\infty}(\mathcal{Z}))$ in the main theorem.

We can omit the Besov regularity w.r.t. some components of U.

Lemma

Assume that $U = (V_1, V_2)$ where $V_1 = (U_1, ..., U_s)$ and $V_2 = (U_{s+1}, \dots, U_n)$. If $\mathcal{B}$ does not depend on V_1 and $G = G(V_1, V_2) = G_1(V_1) + G_2(V_2)$ and G_1 is linear then it is enough to assume $U_1, \dots, U_s \in L^3(\mathcal{X})$ in the main theorem.

Boundary

- Until very recently the studies on energy/entropy conservation for various systems were carried out in the periodic setting or in the whole space.
- It however turns out that an extension to bounded domains is not that strenuous to do, provided proper care is taken of the boundary conditions.

Claude Bardos, P. G., A. Świerczewska-Gwiazda, Edriss S. Titi, Emil Wiedemann, On the Extension of Onsager's Conjecture for General Conservation Laws, *Journal of Nonlinear Science*, 2019.

Claude Bardos, P. G., A. Świerczewska-Gwiazda, Edriss S. Titi, Emil Wiedemann, Onsager's Conjecture in Bounded Domains for the Conservation of Entropy and other Companion Laws, arXiv:1902.07120

Boundary

- The study of sufficient conditions for energy conservation in bounded domains has been undertaken firstly for the incompressible Euler with impermeability boundary condition

$$v \cdot n = 0$$

Consequently the energy flux vanishes on the boundary in the normal direction

$$q(v,p) \cdot n = \left(\frac{1}{2}|v|^2 + p\right) v \cdot n = 0.$$

- C. Bardos, E. Titi. Onsager's Conjecture for the Incompressible Euler Equations in Bounded Domains. Arch. Rational Mech. Anal. 2018

The authors make an assumption on v only: $v \in L^3(0,T;C^\alpha(\overline{\Omega}))$ and later recover from the equation the information on the regularity of the pressure p, i.e. $p \in L^{3/2}(0,T;C^\alpha(\overline{\Omega}))$, what allows to justify the meaning of the above boundary condition point-wise.

Boundary

C. Bardos, E. Titi, and E. Wiedemann. Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit. *Communications in Mathematical Physics*, 2019.

Here the authors relax this assumption, requiring only interior Hölder regularity and continuity of the normal component of the energy flux near the boundary.

Connections to Kolmogorov's theory of turbulence - incompressible Euler equations

According to Kolmogorov the energy spectrum function $E(k)$ in the inertial range in a turbulent flow is given by a power law relation

$$E(k) = C\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}},$$

where k is the modulus of the wave vector corresponding to some harmonics in the Fourier representation of the flow velocity field, and by ε we mean the ensemble average of the energy dissipation rate $\varepsilon = v\langle|\nabla u|^2\rangle$. This relation, stated in physical space, corresponds exactly to the conjecture of Onsager up to the difference that Kolmogorov theory concerns statistically averaged quantities.

Higher order systems

Euler-Korteweg Equations

We now consider the isothermal Euler-Korteweg system in the form

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = -\rho \nabla_x \left(h'(\rho) + \frac{\kappa'(\rho)}{2} |\nabla_x \rho|^2 - \mathrm{div}(\kappa(\rho) \nabla_x \rho) \right),$$

where $\rho \geq 0$ is the scalar density of a fluid, u is its velocity, $h = h(\rho)$ is the energy density and $\kappa(\rho) > 0$ is the coefficient of capillarity.

In conservative form

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = \mathrm{div}\, S,$$
$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

where S is the Korteweg stress tensor

$$S = [-p(\rho) - \frac{\rho \kappa'(\rho) + \kappa(\rho)}{2} |\nabla_x \rho|^2 + \mathrm{div}(\kappa(\rho) \rho \nabla_x \rho)]\mathbb{I} - \kappa(\rho) \nabla_x \rho \otimes \nabla_x \rho$$

where the local pressure is defined as $p(\rho) = \rho h'(\rho) - h(\rho)$.

Energy Equality

It can be shown that smooth solutions to the EK system satisfy the balance of total (kinetic and internal) energy

$$\partial_t \left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{\kappa(\rho)}{2}|\nabla_x\rho|^2 \right)$$
$$+ \operatorname{div} \left(\rho u \left(\frac{1}{2}|u|^2 + h'(\rho) + \frac{\kappa'(\rho)}{2}|\nabla_x\rho|^2 - \operatorname{div}(\kappa(\rho)\nabla_x\rho) \right)\right.$$
$$\left. -\kappa(\rho)\partial_t\rho\nabla\rho \right) = 0.$$

Energy Conservation for Euler-Korteweg equations

Theorem

Let (ρ, u) be a solution to the EK system with constant capillarity in the sense of distributions. Assume

$$u, \nabla_x u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u, \nabla_x \rho, \Delta\rho \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d),$$

where $0 < \alpha, \beta < 1$ *such that* $\min(2\alpha+\beta, \alpha+2\beta) > 1$.
Then the energy is locally conserved, i.e.

$$\partial_t\Big(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{\kappa}{2}|\nabla_x\rho|^2\Big) + \operatorname{div}\Big(\frac{1}{2}\rho u|u|^2 + \rho^2 u - \kappa\rho u\Delta\rho - \kappa\partial_t\rho\nabla\rho\Big) = 0$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

T.Dębiec, P.G., A.Świerczewska-Gwiazda, A.Tzavaras. Conservation of energy for the Euler-Korteweg equations. *Calculus of Variations and*

Some remarks:

1. In 1-D isentropic Euler equation has infinite number of entropies

DiPerna, Ronald J. Convergence of the viscosity method for isentropic gas dynamics *Comm. Math. Phys.,* 1983

Lions, P.-L. and Perthame, B. and Tadmor, E. Kinetic formulation of the isentropic gas dynamics and p-systems *Comm. Math. Phys.,* 1993

2. Averaging contraction principle for scalar conservation laws (measure valued -measure valued uniqueness)

DiPerna, Ronald J. Measure-valued solutions to conservation laws *Arch. Rational Mech. Anal.,* 1985

Szepessy, Anders An existence result for scalar conservation laws using measure valued solutions *Comm. Partial Differential Equations,* 1989

Dębiec, Tomasz, P.G., Łyczek, Kamila and Świerczewska-Gwiazda, Agnieszka, Relative entropy method for measure-valued solutions in natural sciences *Topol. Methods Nonlinear Anal.* 2018

Thank you for your attention