On some non-uniqueness results for the compressible Euler equations based on convex integration

Simon Markfelder

University of Würzburg

August 13, 2019

Workshop "Convex Integration in PDEs, Geometry, and Variational Calculus", Banff, Canada

Isentropic Euler Equations

Isentropic compressible Euler equations

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0 \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) &= 0
\end{aligned}
$$

Isentropic compressible Euler equations

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) &= 0 \end{aligned}$$

Unknowns:

- density $\varrho : [0, T) \times \Omega \to \mathbb{R}^+$
- velocity $\mathbf{u} : [0, T) \times \Omega \to \mathbb{R}^n$

Variables:

- time $t \in [0, T)$
- spatial variable $\mathbf{x} = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $n = 2, 3$

Isentropic compressible Euler equations

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) &= 0 \end{aligned}$$

Unknowns:

- density $\varrho : [0, T) \times \Omega \to \mathbb{R}^+$
- velocity $\mathbf{u} : [0, T) \times \Omega \to \mathbb{R}^n$

Variables:

- time $t \in [0, T)$
- spatial variable $\mathbf{x} = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $n = 2, 3$

The **pressure** $p = p(\varrho)$ is a given function.

Example: Polytropic pressure law

$$p(\varrho) = \varrho^\gamma \quad \text{where } \gamma > 1$$

Initial boundary value problem

Initial condition:

$$\varrho(0,\cdot) = \varrho_0 \quad , \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

Boundary conditions:

- Periodic boundary condition
- Impermeability boundary condition: $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$
- ...

Admissible solutions

We supplement the Euler equations with the *energy inequality*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) + \mathrm{div}\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho)\right)\mathbf{u}\right] \leq 0$$

where $P = P(\varrho)$ is the **pressure potential**.

Admissible solutions

We supplement the Euler equations with the *energy inequality*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho)\right)\mathbf{u}\right] \leq 0$$

where $P = P(\varrho)$ is the **pressure potential**.

Polytropic pressure law

$$P(\varrho) = \frac{1}{\gamma - 1}\varrho^\gamma$$

Admissible solutions

We supplement the Euler equations with the *energy inequality*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho)\right)\mathbf{u}\right] \leq 0$$

where $P = P(\varrho)$ is the **pressure potential**.

Polytropic pressure law

$$P(\varrho) = \frac{1}{\gamma - 1}\varrho^\gamma$$

Definition

A weak solution is called *admissible* if the energy inequality is fulfilled in the sense of distributions.

Convex integration for *incompressible* Euler

De Lellis and Székelyhidi showed existence of infinitely many solutions $(\mathbf{v}, p)$ to the *incompressible* Euler equations

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p &= 0, \end{aligned}$$

where one can prescribe the kinetic energy $\frac{1}{2}|\mathbf{v}(t, \mathbf{x})|^2 = \overline{e}(t, \mathbf{x})$ for a.e. $(t, \mathbf{x})$.

C. De Lellis and L. Székelyhidi Jr. "The Euler equations as a differential inclusion". In: *Ann. of Math. (2)* 170.3 (2009), pp. 1417–1436

C. De Lellis and L. Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: *Arch. Ration. Mech. Anal.* 195.1 (2010), pp. 225–260

Non-uniqueness results

Non-uniqueness results

Theorem

For any pressure function $p(\varrho)$ there exist initial data $(\varrho_0, \mathbf{u}_0)$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{u})$.

C. De Lellis and L. Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: Arch. Ration. Mech. Anal. 195.1 (2010), pp. 225–260

Non-uniqueness results

Theorem

For any pressure function $p(\varrho)$ there exist initial data $(\varrho_0, \mathbf{u}_0)$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{u})$.

C. De Lellis and L. Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: Arch. Ration. Mech. Anal. 195.1 (2010), pp. 225–260

Theorem

For any pressure function $p(\varrho)$ and any given periodic initial density $\varrho_0 \in C^1$ there exist a periodic initial velocity $\mathbf{u}_0 \in L^\infty$ for which there are infinitely many space-periodic admissible weak solutions $(\varrho, \mathbf{u})$.

E. Chiodaroli. "A counterexample to well-posedness of entropy solutions to the compressible Euler system". In: J. Hyperbolic Differ. Equ. 11.3 (2014), pp. 493–519

E. Feireisl. "Maximal dissipation and well-posedness for the compressible Euler system". In: J. Math. Fluid Mech. 16 (2014), pp. 447–461

Riemann problem

Here $\Omega = \mathbb{R}^2$, $T = \infty$.

Write $\mathbf{u} = (v, u)$.

Riemann problem

Here $\Omega = \mathbb{R}^2, \; T = \infty$.

Write $\mathbf{u} = (v, u)$.

Riemann initial data

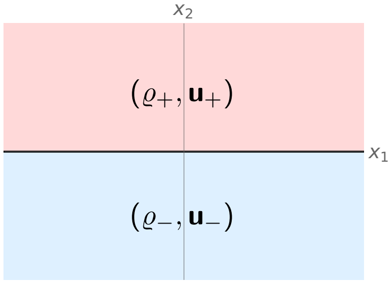
$$(\varrho, \mathbf{u})(0, \mathbf{x}) = (\varrho_0, \mathbf{u}_0)(\mathbf{x}) := \begin{cases} (\varrho_-, \mathbf{u}_-) & \text{if } x_2 < 0 \\ (\varrho_+, \mathbf{u}_+) & \text{if } x_2 > 0 \end{cases},$$

where $\varrho_\pm \in \mathbb{R}^+$ and $\mathbf{u}_\pm \in \mathbb{R}^2$ are constant and $v_- = v_+ = 0$.

Riemann problem

Here $\Omega = \mathbb{R}^2$, $T = \infty$.

Write $\mathbf{u} = (v, u)$.



Riemann initial data

$$(\varrho, \mathbf{u})(0, \mathbf{x}) = (\varrho_0, \mathbf{u}_0)(\mathbf{x}) := \begin{cases} (\varrho_-, \mathbf{u}_-) & \text{if } x_2 < 0 \\ (\varrho_+, \mathbf{u}_+) & \text{if } x_2 > 0 \end{cases},$$

where $\varrho_\pm \in \mathbb{R}^+$ and $\mathbf{u}_\pm \in \mathbb{R}^2$ are constant and $v_- = v_+ = 0$.

Corresponding 1-d Riemann problem

Solve the corresponding 1-d Riemann problem

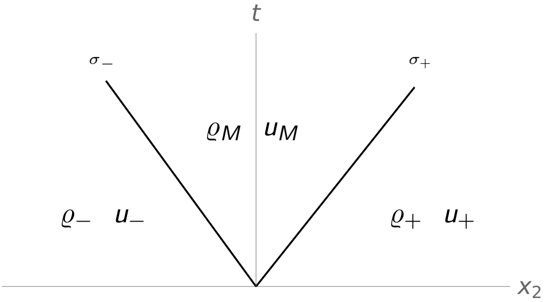
$$
\begin{aligned}
\partial_t \varrho + \partial_{x_2}(\varrho u) &= 0, \\
\partial_t(\varrho u) + \partial_{x_2}\big(\varrho u^2 + p(\varrho)\big) &= 0,
\end{aligned}
$$

$$
(\varrho, u)(0, x_2) = (\varrho_0, u_0)(x_2) := \begin{cases} (\varrho_-, u_-) & \text{if } x_2 < 0 \\ (\varrho_+, u_+) & \text{if } x_2 > 0 \end{cases} .
$$

Solution of the corresponding 1-d Riemann problem

Constant states seperated by two waves

- 1-wave: Either a shock or a rarefaction wave
- 2-wave: Either a shock or a rarefaction wave



Possible structure of the 1-d Riemann solution

1-wave	2-wave	
-	-	
raref.	-	
-	raref.	
raref.	raref.	
shock	shock	
shock	raref.	
raref.	shock	
-	shock	
shock	-	

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	
-	-	
raref.	-	
-	raref.	
raref.	raref.	
shock	shock	
shock	raref.	
raref.	shock	
-	shock	
shock	-	

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	
-	-	✓
raref.	-	✓
-	raref.	✓
raref.	raref.	✓
shock	shock	
shock	raref.	
raref.	shock	
-	shock	
shock	-	

G.-Q. Chen and J. Chen. "Stability of rarefaction waves and vacuum states for the multidimensional Euler equations". In: *J. Hyperbolic Differ. Equ.* 4.1 (2007), pp. 105–122

E. Feireisl and O. Kreml. "Uniqueness of rarefaction waves in multidimensional compressible Euler system". In: *J. Hyperbolic Differ. Equ.* 12.3 (2015), pp. 489–499

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	
-	-	✓
raref.	-	✓
-	raref.	✓
raref.	raref.	✓
shock	shock	✗
shock	raref.	
raref.	shock	
-	shock	
shock	-	

G.-Q. Chen and J. Chen. "Stability of rarefaction waves and vacuum states for the multidimensional Euler equations". In: *J. Hyperbolic Differ. Equ.* 4.1 (2007), pp. 105–122

E. Feireisl and O. Kreml. "Uniqueness of rarefaction waves in multidimensional compressible Euler system". In: *J. Hyperbolic Differ. Equ.* 12.3 (2015), pp. 489–499

E. Chiodaroli and O. Kreml. "On the energy dissipation rate of solutions to the compressible isentropic Euler system". In: *Arch. Ration. Mech. Anal.* 214.3 (2014), pp. 1019–1049

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	
-	-	✓
raref.	-	✓
-	raref.	✓
raref.	raref.	✓
shock	shock	✗
shock	raref.	✗
raref.	shock	✗
-	shock	✗
shock	-	✗

G.-Q. Chen and J. Chen. "Stability of rarefaction waves and vacuum states for the multidimensional Euler equations". In: *J. Hyperbolic Differ. Equ.* 4.1 (2007), pp. 105–122

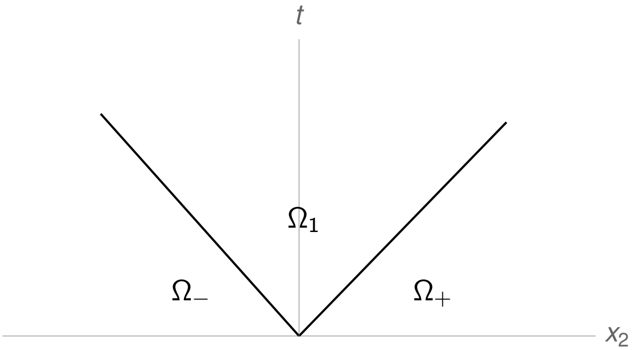
E. Feireisl and O. Kreml. "Uniqueness of rarefaction waves in multidimensional compressible Euler system". In: *J. Hyperbolic Differ. Equ.* 12.3 (2015), pp. 489–499

E. Chiodaroli and O. Kreml. "On the energy dissipation rate of solutions to the compressible isentropic Euler system". In: *Arch. Ration. Mech. Anal.* 214.3 (2014), pp. 1019–1049

C. Klingenberg and S. Markfelder. "The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock". In: *Arch. Ration. Mech. Anal.* 227.3 (2018), pp. 967–994

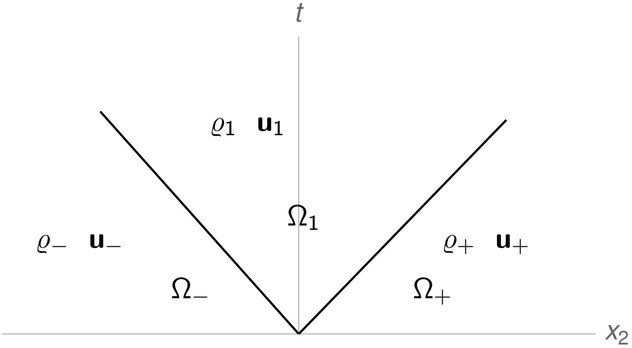
Basic ideas of the non-uniqueness proof

- Definition: fan partition.



Basic ideas of the non-uniqueness proof

- Definition: fan partition.
- Define a piecewise constant *adm. fan subsolution* $(\overline{\varrho}, \overline{\mathbf{u}})$.



Basic ideas of the non-uniqueness proof

- Definition: fan partition.
- Define a piecewise constant *adm. fan subsolution* $(\overline{\varrho}, \overline{\mathbf{u}})$.
- Apply convex integration on Ω_1 to obtain $\widetilde{\mathbf{u}}_1$.

Proposition

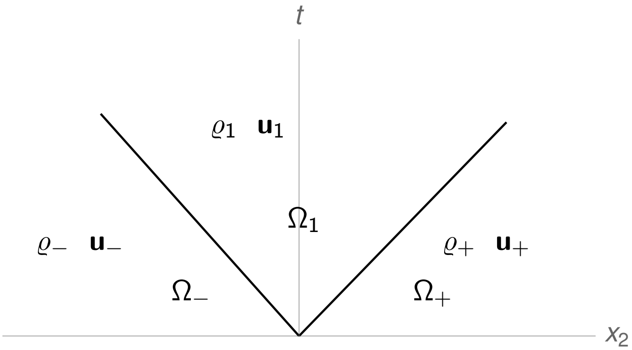
Let $(\mathbf{u}, \mathbb{U}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ *and* $c > 0$ *such that* $\mathbf{u} \otimes \mathbf{u} - \mathbb{U} < \frac{c}{2}\mathbb{I}$. *Furthermore let* $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ *open. Then there exist infinitely many maps* $(\widetilde{\mathbf{u}}, \widetilde{\mathbb{U}}) \in L^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ *with the following properties:*

- $\widetilde{\mathbf{u}}$ *and* $\widetilde{\mathbb{U}}$ *vanish outside* Ω,
- $\operatorname{div} \widetilde{\mathbf{u}} = 0$ *and* $\partial_t \widetilde{\mathbf{u}} + \operatorname{div} \widetilde{\mathbb{U}} = 0$ *in the sense of distributions,*
- $(\mathbf{u} + \widetilde{\mathbf{u}}) \otimes (\mathbf{u} + \widetilde{\mathbf{u}}) - (\mathbb{U} + \widetilde{\mathbb{U}}) = \frac{c}{2}\mathbb{I}$ *a.e. on* Ω.

E. Chiodaroli, C. De Lellis, and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: Comm. Pure Appl. Math. 68.7 (2015), pp. 1157–1190

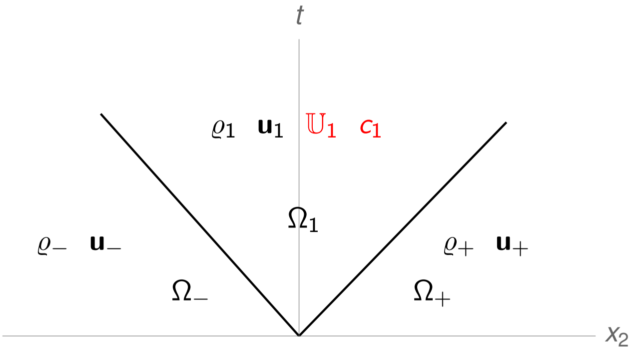
Basic ideas of the non-uniqueness proof

- Definition: fan partition.
- Define a piecewise constant *adm. fan subsolution* $(\overline{\varrho}, \overline{\mathbf{u}})$.
- Apply convex integration on Ω_1 to obtain $\widetilde{\mathbf{u}}_1$.



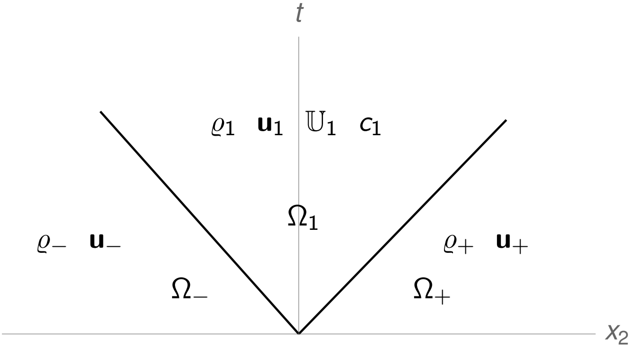
Basic ideas of the non-uniqueness proof

- Definition: fan partition.
- Define a piecewise constant *adm. fan subsolution* $(\overline{\varrho}, \overline{\mathbf{u}}, \overline{\mathbb{U}}, \overline{c})$.
- Apply convex integration on Ω_1 to obtain $\widetilde{\mathbf{u}}_1$.



Basic ideas of the non-uniqueness proof

- Definition: fan partition.
- Define a piecewise constant *adm. fan subsolution* $(\overline{\varrho}, \overline{\mathbf{u}}, \overline{\mathbb{U}}, \overline{c})$.
- Apply convex integration on Ω_1 to obtain $\widetilde{\mathbf{u}}_1$.
- Define the fan subsolution such that $(\overline{\varrho}\,, \overline{\mathbf{u}} + \widetilde{\mathbf{u}}_1)$ is a solution.



Definition: admissible fan subsolution (1)

An adm. fan subsolution consists of 4 piecewise constant functions $(\overline{\varrho}, \overline{\mathbf{u}}, \overline{\mathbb{U}}, \overline{c}) : (0,\infty) \times \mathbb{R}^2 \to (\mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \times \mathbb{R}^+)$, such that:

1. There exists a fan partition $\Omega_-, \Omega_1, \Omega_+$ such that

$$(\overline{\varrho}, \overline{\mathbf{u}}, \overline{\mathbb{U}}, \overline{c}) = \begin{cases} \big(\varrho_\pm\,,\, \mathbf{u}_\pm\,,\, \mathbb{U}_\pm\,,\, c_\pm\big) & \text{on } \Omega_\pm \\ \big(\varrho_1\,,\, \mathbf{u}_1\,,\, \mathbb{U}_1\,,\, c_1\big) & \text{on } \Omega_1 \end{cases}$$

where $\quad \mathbb{U}_\pm = \mathbf{u}_\pm \otimes \mathbf{u}_\pm - \frac{1}{2}|\mathbf{u}_\pm|^2\,\mathbb{I} \quad$ and $\quad c_\pm = |\mathbf{u}_\pm|^2.$

2. The following inequality holds in the sense of definiteness

$$\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbb{U}_1 < \frac{1}{2} c_1\,\mathbb{I}.$$

Definition: admissible fan subsolution (2)

3. The following identities hold in the sense of distributions:

$$
\begin{aligned}
\partial_t \overline{\varrho} + \operatorname{div}(\overline{\varrho}\, \overline{\mathbf{u}}) &= 0, \\
\partial_t (\overline{\varrho}\, \overline{\mathbf{u}}) + \operatorname{div}(\overline{\varrho}\, \overline{\mathbb{U}}) + \nabla \left(p(\overline{\varrho}) + \frac{1}{2} \overline{\varrho}\, \overline{c} \right) &= 0.
\end{aligned}
$$

4. The energy inequality is fulfilled in the sense of distributions:

$$
\partial_t \left(\frac{1}{2} \overline{\varrho}\, \overline{c} + P(\overline{\varrho}) \right) + \operatorname{div} \left[\left(\frac{1}{2} \overline{\varrho}\, \overline{c} + P(\overline{\varrho}) + p(\overline{\varrho}) \right) \overline{\mathbf{u}} \right] \leq 0.
$$

Condition for the existence of infinitely many solutions

Proposition

Existence of an admissible fan subsolution $\Longrightarrow$ *Existence of infinitely many adm. weak solutions*

E. Chiodaroli, C. De Lellis, and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: Comm. Pure Appl. Math. 68.7 (2015), pp. 1157–1190

Condition for the existence of infinitely many solutions

Proposition

Existence of an admissible fan subsolution $\Longrightarrow$ *Existence of infinitely many adm. weak solutions*

E. Chiodaroli, C. De Lellis, and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: Comm. Pure Appl. Math. 68.7 (2015), pp. 1157–1190

Translate the definition of an admissible fan subsolution into a system of algebraic equations and inequalities.

Condition for the existence of infinitely many solutions

Proposition

Existence of an admissible fan subsolution $\Longrightarrow$ *Existence of infinitely many adm. weak solutions*

E. Chiodaroli, C. De Lellis, and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: Comm. Pure Appl. Math. 68.7 (2015), pp. 1157–1190

Translate the definition of an admissible fan subsolution into a system of algebraic equations and inequalities.

More precisely we obtain a system of **6 equations** and **4 inequalities** for **8 unknowns**.

Solution to the algebraic equations and inequalities

Two shocks:

Solution to the algebraic equations and inequalities

Two shocks:

- Small perturbation of the 1-d Riemann solution yields a solution to the algebraic equations and inequalities.

E. Chiodaroli and O. Kreml. "On the energy dissipation rate of solutions to the compressible isentropic Euler system". In: *Arch. Ration. Mech. Anal.* 214.3 (2014), pp. 1019–1049

Solution to the algebraic equations and inequalities

One shock, one rarefaction:

- First notice that if the rarefaction is “small” then there is a solution to the algebraic equations and inequalities.

Solution to the algebraic equations and inequalities

One shock, one rarefaction:

- First notice that if the rarefaction is “small” then there is a solution to the algebraic equations and inequalities.
- Introduce a rarefaction to an intermediate state to obtain such a small rarefaction.

Solution to the algebraic equations and inequalities

One shock, one rarefaction:

- First notice that if the rarefaction is "small" then there is a solution to the algebraic equations and inequalities.
- Introduce a rarefaction to an intermediate state to obtain such a small rarefaction.

A single shock:

- Introduce a shock wave to an intermediate state to obtain a small rarefaction.

C. Klingenberg and S. Markfelder. "The Riemann problem for the multidimensional isentropic system of gas dynamics is ill-posed if it contains a shock". In: *Arch. Ration. Mech. Anal.* 227.3 (2018), pp. 967–994

Further results

Theorem

There exist Lipschitz continuous initial data $(\varrho_0, \mathbf{u}_0)$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{u})$.

E. Chiodaroli, C. De Lellis, and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: Comm. Pure Appl. Math. 68.7 (2015), pp. 1157–1190

Further results

Theorem

There exist Lipschitz continuous initial data $(\varrho_0, \mathbf{u}_0)$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{u})$.

E. Chiodaroli, C. De Lellis, and O. Kreml. "Global ill-posedness of the isentropic system of gas dynamics". In: Comm. Pure Appl. Math. 68.7 (2015), pp. 1157–1190

Theorem

There exist Riemann initial data $(\varrho_0, \mathbf{u}_0)$ for which there are infinitely many energy-conserving weak solutions $(\varrho, \mathbf{u})$.

C. Klingenberg and S. Markfelder. "Non-uniqueness of energy-conservative solutions to the isentropic compressible two-dimesnsional Euler equations". In: J. Hyperbolic Differ. Equ. 15.4 (2018), pp. 721–730

Full Euler Equations

Compressible Euler equations

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) &= 0 \\
\partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) &= 0 \\
\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div} \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right) \mathbf{u} \right] &= 0
\end{aligned}
$$

Compressible Euler equations

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0 \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) &= 0 \\
\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \operatorname{div} \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right) \mathbf{u} \right] &= 0
\end{aligned}
$$

Unknowns:

- density $\varrho : [0, T) \times \Omega \to \mathbb{R}^+$
- velocity $\mathbf{u} : [0, T) \times \Omega \to \mathbb{R}^n$
- temperature $\vartheta : [0, T) \times \Omega \to \mathbb{R}^+$

Variables:

- time $t \in [0, T)$
- spatial variable $\mathbf{x} = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $n = 2, 3$

Compressible Euler equations

$$\begin{aligned}
\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) &= 0 \\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) &= 0 \\
\partial_t \left(\tfrac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div} \left[\left(\tfrac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right) \mathbf{u} \right] &= 0
\end{aligned}$$

The **pressure** $p = p(\varrho, \vartheta)$ and the **internal energy** $e = e(\varrho, \vartheta)$ are given functions.

Example (Ideal gas)

$$\begin{aligned}
p(\varrho, \vartheta) &= \varrho \vartheta \\
e(\varrho, \vartheta) &= \tfrac{1}{\gamma - 1} \vartheta \qquad \text{where } \gamma > 1 \text{ (adiabatic exponent)}
\end{aligned}$$

Admissible solutions

Definition

A weak solution is called *admissible* if the entropy inequality

$$\partial_t\Big(\varrho\, s(\varrho,\vartheta)\Big) + \operatorname{div}\Big(\varrho\, s(\varrho,\vartheta)\,\mathbf{u}\Big) \geq 0,$$

is fulfilled in the sense of distributions.

Here $s(\varrho,\vartheta)$ is the **entropy**, e. g. for the ideal gas

$$s(\varrho,\vartheta) = \tfrac{1}{\gamma-1}\log\vartheta - \log\varrho.$$

Non-uniqueness results for the full Euler equations

Non-uniqueness results for the full Euler equations

Theorem

For any given piecewise-constant initial density ϱ_0 and temperature ϑ_0 there exists an initial velocity $\mathbf{u}_0 \in L^\infty$ for which there are infinitely many admissible weak solutions $(\varrho, \mathbf{u}, \vartheta)$.

E. Feireisl, C. Klingenberg, O. Kreml, and S. Markfelder. On oscillatory solutions to the complete Euler system. submitted. 2017. arXiv: 1710.10918

see also:
T. Luo, C. Xie, and Z. Xin. "Non-uniqueness of admissible weak solutions to compressible Euler systems with source terms". In: Adv. Math. 291 (2016), pp. 542–583

Riemann problem

Consider *ideal gas*, i. e.

$$p(\varrho, \vartheta) = \varrho\vartheta,$$
$$e(\varrho, \vartheta) = \tfrac{1}{\gamma - 1}\vartheta.$$

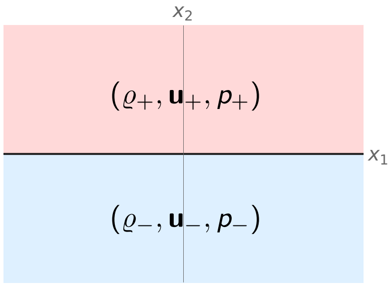
It is more convenient to consider
p as an unknown instead of ϑ.

Riemann problem

Consider *ideal gas*, i. e.

$$p(\varrho, \vartheta) = \varrho\vartheta,$$
$$e(\varrho, \vartheta) = \frac{1}{\gamma - 1}\vartheta.$$

It is more convenient to consider p as an unknown instead of ϑ.



Riemann initial data

$$(\varrho, \mathbf{u}, p)(0, \mathbf{x}) = (\varrho_0, \mathbf{u}_0, p_0)(\mathbf{x}) := \begin{cases} (\varrho_-, \mathbf{u}_-, p_-) & \text{if } x_2 < 0 \\ (\varrho_+, \mathbf{u}_+, p_+) & \text{if } x_2 > 0 \end{cases},$$

where $\varrho_\pm \in \mathbb{R}^+$, $\mathbf{u}_\pm \in \mathbb{R}^2$ and $p_\pm \in \mathbb{R}^+$ are constant and $v_- = v_+ = 0$ (remember that $\mathbf{u} = (v, u)$).

Corresponding 1-d Riemann problem

Solve the corresponding 1-d Riemann problem

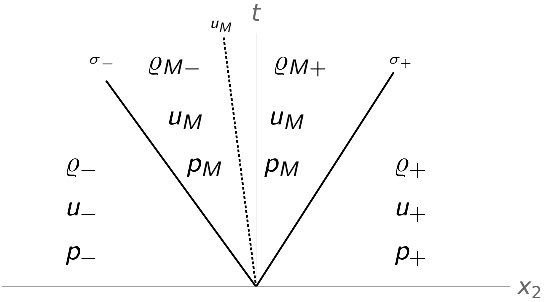
$$\begin{aligned} \partial_t \varrho + \partial_{x_2}(\varrho u) &= 0, \\ \partial_t(\varrho u) + \partial_{x_2}\left(\varrho u^2 + p\right) &= 0, \\ \partial_t\left(\frac{1}{2}\varrho u^2 + \varrho e(\varrho, p)\right) + \partial_{x_2}\left[\left(\frac{1}{2}\varrho u^2 + \varrho e(\varrho, p) + p\right) u\right] &= 0, \end{aligned}$$

$$(\varrho, u, p)(0, x_2) = (\varrho_0, u_0, p_0)(x_2) := \begin{cases} (\varrho_-, u_-, p_-) & \text{if } x_2 < 0 \\ (\varrho_+, u_+, p_+) & \text{if } x_2 > 0 \end{cases} .$$

Solution of the corresponding 1-d Riemann problem

Constant states seperated by three waves

- 1-wave: Either a shock or a rarefaction wave
- 2-wave: Contact discontinuity
- 3-wave: Either a shock or a rarefaction wave



Possible structure of the 1-d Riemann solution

1-wave	2-wave	3-wave	
-	-	-	
-	-	shock	
-	-	raref.	
shock	-	-	
shock	-	shock	
shock	-	raref.	
raref.	-	-	
raref.	-	shock	
raref.	-	raref.	

1-wave	2-wave	3-wave	
-	contact	-	
-	contact	shock	
-	contact	raref.	
shock	contact	-	
shock	contact	shock	
shock	contact	raref.	
raref.	contact	-	
raref.	contact	shock	
raref.	contact	raref.	

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	3-wave	
-	-	-	
-	-	shock	
-	-	raref.	
shock	-	-	
shock	-	shock	
shock	-	raref.	
raref.	-	-	
raref.	-	shock	
raref.	-	raref.	

1-wave	2-wave	3-wave	
-	contact	-	
-	contact	shock	
-	contact	raref.	
shock	contact	-	
shock	contact	shock	
shock	contact	raref.	
raref.	contact	-	
raref.	contact	shock	
raref.	contact	raref.	

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	3-wave	
-	-	-	✓
-	-	shock	
-	-	raref.	✓
shock	-	-	
shock	-	shock	
shock	-	raref.	
raref.	-	-	✓
raref.	-	shock	
raref.	-	raref.	✓

1-wave	2-wave	3-wave	
-	contact	-	
-	contact	shock	
-	contact	raref.	
shock	contact	-	
shock	contact	shock	
shock	contact	raref.	
raref.	contact	-	
raref.	contact	shock	
raref.	contact	raref.	

Possible structure of the 1-d Riemann solution

Question

Is the 1-d Riemann solution the *unique* adm. weak solution?

1-wave	2-wave	3-wave	
-	-	-	✓
-	-	shock	
-	-	raref.	✓
shock	-	-	
shock	-	shock	✗
shock	-	raref.	
raref.	-	-	✓
raref.	-	shock	
raref.	-	raref.	✓

1-wave	2-wave	3-wave	
-	contact	-	
-	contact	shock	
-	contact	raref.	
shock	contact	-	
shock	contact	shock	✗
shock	contact	raref.	
raref.	contact	-	
raref.	contact	shock	
raref.	contact	raref.	

References

Uniqueness:

G.-Q. Chen and J. Chen. "Stability of rarefaction waves and vacuum states for the multidimensional Euler equations". In: *J. Hyperbolic Differ. Equ.* 4.1 (2007), pp. 105–122

E. Feireisl, O. Kreml, and A. Vasseur. "Stability of the isentropic Riemann solutions of the full multidimensional Euler system". In: *SIAM J. Math. Anal.* 47.3 (2015), pp. 2416–2425

Non-uniqueness:

S. Markfelder et al. *Non-uniqueness of admissible weak solutions to the Riemann problem for the full Euler system in 2D*. submitted. 2018. arXiv: 1805.11354

Idea of the non-uniqueness proof

- Define a piecewise constant *admissible fan subsolution* as in the isentropic case, but with 4 sectors instead of 3.

Idea of the non-uniqueness proof

- Define a piecewise constant *admissible fan subsolution* as in the isentropic case, but with 4 sectors instead of 3.
- $$\begin{array}{c}\text{Existence of an}\\ \text{admissible fan subsolution}\end{array} \Longrightarrow \begin{array}{c}\text{Existence of infinitely}\\ \text{many adm. weak solutions}\end{array}$$

Idea of the non-uniqueness proof

- Define a piecewise constant *admissible fan subsolution* as in the isentropic case, but with 4 sectors instead of 3.
- $$\begin{array}{c}\text{Existence of an}\\ \text{admissible fan subsolution}\end{array} \Longrightarrow \begin{array}{c}\text{Existence of infinitely}\\ \text{many adm. weak solutions}\end{array}$$
- The definition of an admissible fan subsolution can be translated into a system of algebraic equations and inequalities.

Idea of the non-uniqueness proof

- Define a piecewise constant *admissible fan subsolution* as in the isentropic case, but with 4 sectors instead of 3.
- $$\begin{matrix}\text{Existence of an} \\ \text{admissible fan subsolution}\end{matrix} \Longrightarrow \begin{matrix}\text{Existence of infinitely} \\ \text{many adm. weak solutions}\end{matrix}$$
- The definition of an admissible fan subsolution can be translated into a system of algebraic equations and inequalities.
- As in the isentropic case, a small perturbation of the 1-d Riemann solution yields a solution to the algebraic equations an inequalities.

S. Markfelder et al. *Non-uniqueness of admissible weak solutions to the Riemann problem for the full Euler system in 2D*. submitted. 2018. arXiv: 1805.11354