# Ill-posedness in fluid dynamics

# – what can we do about it?

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based on a joint works with D. Breit and E. Feireisl

$$\begin{aligned} \partial_t \varrho + \operatorname{div} m &= 0, \\ \partial_t m + \operatorname{div}\left(\frac{m \otimes m}{\varrho}\right) + \nabla \varrho^\gamma &= 0, \end{aligned} \qquad x \in \mathbb{T}^d, t \in (0,T),$$

$$\varrho(0) = \varrho_0, \qquad m(0) = m_0$$

- density $\varrho \colon [0,T] \times \mathbb{T}^d \to [0,\infty)$
- momentum $m \colon [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, corresponds to $m = \varrho u$ where $u$ is velocity
- $\varrho^\gamma$ pressure, $\gamma > 1$ adiabatic constant
- $d = 2, 3$

**Existence? Uniqueness?**

- strong solutions exist only locally
- shocks appear
- weak solutions not unique

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div} m &= 0,\\
\partial_t m + \operatorname{div}\left(\frac{m\otimes m}{\varrho}\right) + \nabla \varrho^\gamma &= G(\varrho, m)\frac{\mathrm{d}W}{\mathrm{d}t},
\end{aligned}
$$

- either $G(\varrho,m)=\varrho G(x)$ or $G(\varrho,m)=m$
- **Brownian motion** $W\colon \Omega\times[0,T]\to\mathbb{R}$ – only $C^\alpha([0,T])$ trajectories for $\alpha<1/2$
- probability needed to make sense of the stochastic forcing
- good (reasonable) solutions shall be measurable wrt the noise – **adapted**
    - $(\varrho,m)|_{[0,t]}\colon \Omega\to C_{\mathrm{weak}}\Big([0,t];L^\gamma\times L^{\frac{2\gamma}{\gamma+1}}\Big)$ measurable wrt $\sigma(W|_{[0,t]})$
    - probabilistically **strong** solutions
    - typically do not exist for problems without uniqueness
    - existence by compactness $\Rightarrow$ probabilistically **weak** solutions

**Theorem (Breit, Feireisl, H., APDE '19)** *Let $T > 0$. Let $[\varrho_0, m_0] \in C^3$, $\mathcal{F}_0$-measurable, $\varrho_0 > 0$ a.s.*

*There exist stopping times $\tau_M > 0$ such that $\tau_M \uparrow \infty$ a.s. and for every $M > 0$*

*the system admits infinitely many adapted weak solutions on $[0, \tau_M \wedge T]$.*

- **strong** in the probabilistic sense, **weak** in the PDE sense
- stochastic version of the oscillatory lemma à la De Lellis–Székelyhidi, Feireisl
  - rewrite as an abstract Euler system
  - reduce to the oscillatory lemma in the incompressible setting
  - keep track of the arrow of time – added oscillations are adapted

**Search for physical solutions**

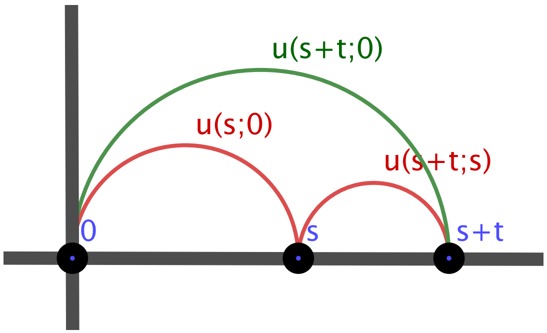
- multiple weak solutions emanating from the same initial data
  - **admissibility criterium** needed to select the physical one
- **energy balance** (in a suitable e.g. integrated/weak form)

$$e := \frac{1}{2}\frac{|m|^2}{\varrho} + \frac{1}{\gamma-1}\varrho^\gamma$$

$$\partial_t e + \operatorname{div}\left[(e+\varrho^\gamma)\frac{m}{\varrho}\right] \leqslant 0$$

- convex integration by De Lellis–Székelyhidi, Chiodaroli et al.
  - **infinitely many** admissible weak solutions (even for certain smooth initial data)
- additional selection criterium à la Dafermos
  - **maximality of energy dissipation** – the energy is dissipated at highest possible rate
  - remains ill-posed

**A physical property implied by uniqueness** - semiflow property




- starting from $0$ and going to $s+t$ gives the same output as $0 \to s \to s+t$
- very unclear if uniqueness not valid

**Question:** existence of a **semiflow selection**?

- for an initial time $s$ there are possibly multiple solutions
- **choose one** of them so that the semiflow property holds
- this would give a selection of better behaved (more physical) solutions

**Theorem (Breit, Feireisl, H., ARMA '19)** *The Euler system admits a solution semiflow in the class of admissible dissipative solutions (minimizing the total energy).*

**Physical relevance of the selection justified through:**

- stability of strong solutions
  - strong solutions are unique (in the class of dissipative solutions) – are always selected
- maximal dissipation of energy
  - the selected solution is admissible
- stability of stationary states $\varrho(T,\cdot) \equiv \mathrm{const}$, $m(T,\cdot) \equiv 0$
  - if a stationary state is reached, the system remains there (because of energy minimization)
- wild solutions by convex integration are ruled out (at least to some extent)

## Dissipative solutions

*Consider an approximation (e.g. vanishing viscosity limit)*

$$\begin{aligned} \partial_t \varrho_n + \operatorname{div} m_n &= F_{1,n} & \qquad \varrho_n(0) = \varrho_{n,0} \\ \partial_t m_n + \operatorname{div}\left(\frac{m_n \otimes m_n}{\varrho_n}\right) + \nabla \varrho_n^\gamma &= F_{2,n} & \qquad m_n(0) = m_{n,0} \end{aligned}$$

*with* $F_{1,n}, F_{2,n} \to 0$ *in* $\mathcal{D}'((0,T) \times \mathbb{T}^d)$.

- the energy inequality – the only source of a priori estimates (needed for the approximation)

$$\int \left[\frac{1}{2}\frac{|m_n|^2}{\varrho_n} + \frac{1}{\gamma-1}\varrho_n^\gamma\right](t,x)\,\mathrm{d}x \leqslant \int \left[\frac{1}{2}\frac{|m_{n,0}|^2}{\varrho_{n,0}} + \frac{1}{\gamma-1}\varrho_{n,0}^\gamma\right](x)\,\mathrm{d}x \leqslant E_0$$

  implies uniform bounds $\varrho_n \in L^\infty(0,T;L^\gamma)$ and $m_n \in L^\infty\left(0,T;L^{\frac{2\gamma}{\gamma+1}}\right)$
- hence weak convergence $\varrho_n \xrightarrow{w} \varrho$ and $m_n \xrightarrow{w} m$

**Can we pass to the limit in the equation?**

- no compactness – cannot pass to the limit in the nonlinearities
    - oscillations, concentrations

- but maybe there is a hidden regularity?

**Theorem (Breit, Feireisl, H. '19)** *Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $\varrho_0 \in L^\infty$, $\varrho_0 > 0$. There exists a sequence of weak solutions $[\varrho_n, m_n]$ to the Euler system with $\varrho_n = \varrho_n(x)$ such that*

$$\varrho_n \xrightarrow{w*} \varrho_0 \text{ in } L^\infty(D), \qquad m_n \xrightarrow{w*} 0 \text{ in } L^\infty((0,T) \times D),$$

$$\liminf_{n \to \infty} \| \varrho_n - \varrho \|_{L^1} > 0.$$

- only choosing $\varrho_0 \equiv \mathrm{const}$ gives a weak solution in the limit
- otherwise the limit is not a weak solution

**Theorem (Feireisl, H. '19)** *Consider a vanishing viscosity approximation of the Euler system on $\mathbb{R}^d$ (with energy inequality) so that*

$$\varrho_n \to \varrho \qquad \text{and} \qquad m_n \to m \qquad \text{in} \qquad \mathcal{D}'((0,T)\times\mathbb{R}^d).$$

*Then either*

- *the convergence is strong in the energy norm*

*or*

- *the limit is not a weak solution to the Euler system.*

- **weak convergence to weak solutions impossible!**
- weak limits are dissipative solutions
- on domains one needs to assume that the convergence is **nicer** at the boundary

**Main ingredient:**

- structure of the system – convexity of the energy and the pressure
- not true in the incompressible case!

They satisfy

$$\begin{aligned} \partial_t\varrho + \operatorname{div} m &= 0 \\ \partial_t m + \operatorname{div}\left(\frac{m\otimes m}{\varrho}\right) + \nabla \varrho^\gamma &= -\operatorname{div}\left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I}\right) \\ \partial_t \int \left[\frac{1}{2}\frac{|m|^2}{\varrho} + \frac{1}{\gamma-1}\varrho^\gamma + \frac{1}{2}\operatorname{tr}[\mathfrak{R}_v] + \frac{1}{\gamma-1}\mathfrak{R}_p\right] \mathrm{d}x &\leqslant 0 \end{aligned}$$

in the sense of distributions with some turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \qquad \mathfrak{R}_p \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}^d)).$$

**A sanity check:** not every $(\varrho, m)$ can be a solution to Euler!

**For approximations:** $\varrho_n \xrightarrow{w} \varrho$ and $m_n \xrightarrow{w} m$

$$\mathfrak{R}_v := \lim_{n\to\infty} \frac{m_n\otimes m_n}{\varrho_n} - \frac{m\otimes m}{\varrho}, \qquad \mathfrak{R}_p := \lim_{n\to\infty} \varrho_n^\gamma - \varrho^\gamma$$

**Proof of the theorem:** limit is a weak solution $\Rightarrow$ $\operatorname{div}\left(\mathfrak{R}_v + \mathfrak{R}_p\mathbb{I}\right) = 0$ in $\mathcal{D}'$

$\Rightarrow$ $\mathfrak{R}_v = 0,\ \mathfrak{R}_p = 0$ $\Rightarrow$ the convergence is strong

**Construction of the semiflow**

**Semiflow:** determined by a triple $[\varrho, m, E]$ with

$$E(t) = \int \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + \frac{1}{\gamma - 1} \varrho^\gamma + \frac{1}{2} \operatorname{tr}[\mathfrak{R}_v] + \frac{1}{\gamma - 1} \mathfrak{R}_p \right] (t, x) \, \mathrm{d}x \qquad a.e. \, t \in (0, \infty)$$

**Admissibility** – $[\varrho^1, m^1, E^1] \prec [\varrho^2, m^2, E^2] \Leftrightarrow E^1(t \pm) \leqslant E^2(t \pm)$ for all $t \in (0, \infty)$

- a solution is admissible $=$ minimal wrt $\prec$

- construction based on ideas from Markov selections (Krylov, Cardona–Kapitanski)
- subsequent minimization/maximization of a sequence of functionals

$$I_{\lambda, F}[\varrho, m, E] = \int_0^\infty e^{-\lambda t} F([\varrho, m, E](t)) \mathrm{d}t$$

- needed: existence, compactness, stability, shift, continuation

**Uniqueness and further properties of the selection?**

- **Existence:** exist globally in time
- **Stability:** weak limits of dissipative solutions are dissipative solutions
- **Weak–strong uniqueness:** satisfied
- **Energy dissipation:** the energy is nonincreasing
- **Semiflow:** maximizes the energy dissipation, rules out wild solutions
- **Consistency:** if continuously differentiable $\Rightarrow$ then they are strong (classical) solutions

**Thanks for your attention!**