Maximal Rigid Subalgebras of Deformations and L^2 Cohomology, II

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BIRS Classification in von Neumann Algebras
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von Neumann Algebras

Definition (Popa)

Let (M, τ) be a finite von Neumann algebra.An s-malleable deformation of M is a tuple $(\alpha_t, \beta, \widetilde{M}, \widetilde{\tau})$ s.t.

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$Q \leq M$ is rigid wrt α if

$$\epsilon_t(Q) := \sup_{x \in (Q)_1} \|\alpha_t(x) - x\|_2 \quad \text{has} \quad \epsilon_t(Q) \to 0 \quad \text{as} \quad t \to 0.$$

We write $Q \in \mathrm{Rig}(\alpha)$

Normalizer, Quasinormalizers, and the Intertwining space

Let $Q \leq M$ be a unital inclusion of tracial von Neumann algebras.

- $\mathcal{N}_M(Q) := \{u \in \mathcal{U}(M) : u^* Q u \subseteq Q\}$
- $\mathcal{N}_M^{wq}(Q) := \{u \in \mathcal{U}(M) : u^* Q u \cap Q \text{ diffuse}\}$
- $\mathcal{Q}^1\mathcal{N}_M(Q) := \left\{x \in M : \exists\, x_1, \ldots, x_k \in M \,\text{s.t.}\, Qx \subseteq \sum_{j=1}^k x_j Q\right\}$

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When Q is diffuse (Λ infinite),

$$W^*(\mathcal{N}_M(Q)) \subseteq W^*(\mathcal{N}_M^{wq}(Q)), W^*(\mathcal{Q}^1\mathcal{N}_M(Q)) \subseteq W^*(wI_M(Q,Q))$$

Rigidity for Normalizers

We assume $M \leq \widetilde{M}$ is an s-malleable deformation of M s.t. $L^2(\widetilde{M}) \ominus L^2(M)$ is a mixing M-M bimodule.

Theorem (Peterson 09)

If $Q \leq M$ is diffuse and $Q \in \mathrm{Rig}(\alpha)$, then $W^*(\mathcal{N}_M(Q))$ is rigid.

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Theorem (ds-Hayes-Hoff-Sinclair 19)

If $Q \leq M$ is diffuse and $Q \in \mathrm{Rig}(\alpha)$, then $W^*(\mathcal{N}_M(Q)), W^*(\mathcal{N}_M^{wq}(Q)), W^*(\mathcal{Q}^1\mathcal{N}_M(Q)), W^*(wI(Q,Q)) \in \mathrm{Rig}(\alpha)$

Estimates on Rates of Convergence

Proposition

Let (M, τ) be a tracial von Neumann algebra and $\alpha_t : \widetilde{M} \to \widetilde{M}$ an s-malleable deformation of (M, τ).
Then for any diffuse α-rigid $Q \leq M$ with $L^2(\widetilde{M}) \ominus L^2(M)$ mixing as a M-M bimodule, the quantities

$$\varepsilon_t(Q) = \sup_{x \in (Q)_1} \|\alpha_t(x) - x\|_2$$

$$\delta_t(Q) = \inf\{\|u - 1\|_2 : u \in \mathcal{U}(\widetilde{M}),\ u^*\alpha_t(x)u = x \text{ for all } x \in Q\}, \text{ and}$$

$$\gamma_t(Q) = \inf\{\|u - 1\|_2 : u \in \mathcal{U}(\widetilde{M}),\ u^*\alpha_t(Q)u \subseteq M\}$$

satisfy

$$\frac{1}{4}\varepsilon_{2t}(Q) \leq \gamma_t(Q) \leq \delta_t(Q) \leq 6\varepsilon_t(Q).$$

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It suffices to show that $W^*(\mathcal{H}_s)$ is rigid where

$$\mathcal{H}_s := \bigcap_{\substack{T \in B(L^2(M), L^2(\tilde{M}) \ominus L^2(M)) \\ T \ Q-Q\text{bimodular}}} \ker(T) \subseteq L^2(M)$$

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Fix $t \in \mathbb{R}$. For any $u \in \mathcal{U}(\tilde{M})$ with $u^* \alpha_t(x) u = x \, \forall \, x \in Q$,
$\Theta_t(x) := u^* \alpha_t(x) u$ satisfies:

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$\Theta_t(\mathcal{H}_s) \subseteq L^2(M) \implies \Theta_t(W^*(\mathcal{H}_s)) \subseteq M$.

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$\Theta_t(\mathcal{H}_s) \subseteq L^2(M) \implies \Theta_t(W^*(\mathcal{H}_s)) \subseteq M$.
Thus $\gamma_t(W^*(\mathcal{H}_s)) \leq \delta_t(Q) \to 0$

L^2 Rigidity

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L^2-rigidity of $L(G)$ is preserved under measure equivalence, implies superrigidity of Bernoulli shifts.

Definition (Peterson-Sinclair 11)

Let (M, τ) be a tracial von Neumann algebra and $Q \leq M$. Q is *L^2-rigid* if for every $(M \subseteq N, \tau_N)$ any tracial von Neumann algebra N and any closeable, real derivation $\delta : L^2(N) \to \mathcal{H}$ with ${}_M\mathcal{H}_M$ embeddable into $(L^2(M) \otimes L^2(M))^{\oplus\infty}$, the $\varphi_t = \exp(-t\delta^*\overline{\delta})$,

$$\sup_{x \in (Q)_1} \|\varphi_t(x) - x\|_2 \to 0$$

as $t \to 0$.

Dilations to Deformations

Definition

Fix (M, τ_M) and let $(\varphi_t)_{t \geq 0}$ be a pointwise-strongly continuous one-parameter semigroup of trace-preserving u.c.p. maps. (φ_t) admits an *s-malleable dilation* $(\widetilde{M}, \alpha, \beta)$, (α_t, β) is an s-malleable deformation of $M \leq \widetilde{M}$ such that

$$\varphi_t(x) \approx \mathbb{E}_M(\alpha_t(x))$$

for all $x \in M$ and $t \in \mathbb{R}$

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Theorem (Dabrowski 10, Junge-Ricard-Shlyakhtenko)

Let M be a II_1 factor and let $\delta : M \to [L^2(M) \otimes L^2(M)]^{\oplus\infty}$ be a closeable, real derivation. Then $\exp(-t\delta^*\overline{\delta})$ admits an s-malleable dilation $(\widetilde{M}, \alpha, \beta)$ so that $L^2(\bigvee_{t\in[0,\infty)} \alpha_t(M)) \ominus L^2(M)$ embeds $(L^2(M) \otimes L^2(M))^{\oplus\infty}$.

Approximate L^2-Rigidity

Definition

Fix (M, τ). $Q \leq M$ is *approximately L^2-rigid* if there exists (Q_n) increasing with $Q_n \leq p_n M p_n$ L^2-rigid and $Q = \bigvee_n Q_n$. $(Q_n)_n$ as a *L^2-rigid filtration* of Q.

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Analog of reduced ℓ^2 cohomology.

Theorem (dS, Hayes, Hoff, Sinclair 19)

$Q \leq M$ is approximately L^2-rigid if and only if $Q = A \oplus (\bigoplus Q_n)$ where A is amenable and Q_n is L^2-rigid.

On a Conjecture of Peterson-Thom 11

Proposition (Peterson-Thom 11)

If G is a discrete group and $H_1, H_2 < G$ with $|H_1 \cap H_2| = \infty$, $H_1 \vee H_2$ has vanishing first reduced ℓ^2-cohomology if both H_1 and H_2 do.

Conjecture

Let M be a II_1 factor and $Q_1, Q_2 \leq M$ such that $Q_i \leq M$ is approximately L^2-rigid for $i = 1, 2$. If $Q_1 \cap Q_2$ is diffuse, then $Q_1 \vee Q_2 \leq M$ is approximately L^2-rigid.

Conjecture (Peterson-Thom)

If $Q_1, Q_2 \leq L(\mathbb{F}_2)$ are amenable and $Q_1 \cap Q_2$ is diffuse, then $Q_1 \vee Q_2$ is amenable.

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`https://math.msu.edu/~nelso787/conferences/GOALS.html`

Thanks!